ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

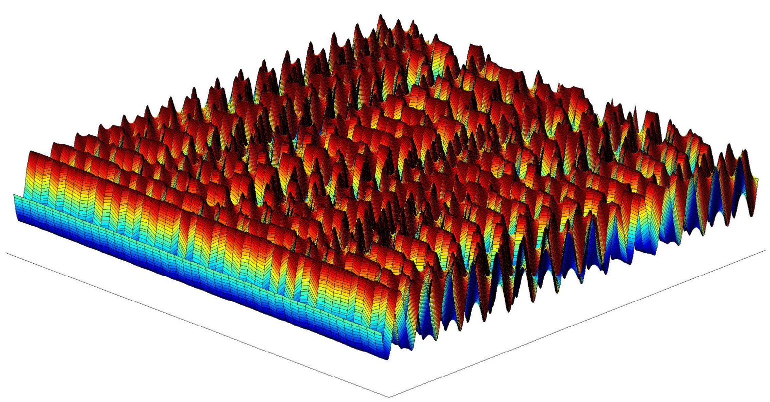
BIOLOGICALLY INSPIRED
ROBOTICS GROUP (BIRG)

Robustness in coupled population of oscillators

Jerome Guerra

February 18, 2005

Master Thesis, winter 2004-2005
Computer and Communication Sciences



Supervisor: Jonas Buchli, BIRG EPFL
Prof. resp: Auke Jan Ijspeert, BIRG EPFL

Contents

1 Inspiration of the project **2**
- 1.1 Introduction . . . 2
- 1.2 Related works . . . 3
- 1.3 Synchronization . . . 4
  - 1.3.1 Non-linear oscillators . . . 4
  - 1.3.2 Synchronization of two coupled oscillators . . . 5
  - 1.3.3 Synchronization in a population of oscillators . . . 6

2 General description of the population of oscillators **8**
- 2.1 Introduction . . . 8
- 2.2 Oscillators, coupling and output . . . 8
  - 2.2.1 Oscillators : amplitude controlled phase oscillator vs Van der Pol's oscillator . . . 9
  - 2.2.2 System's coupling : neighborhood coupled . . . 10
  - 2.2.3 System's output : mean field . . . 11
- 2.3 Implementation and numerical simulations . . . 12
- 2.4 Perturbation's model . . . 13
  - 2.4.1 Thermal noise . . . 13
  - 2.4.2 Faults of fabrication . . . 14
  - 2.4.3 Component aging . . . 15
  - 2.4.4 Variation of initial conditions . . . 15
  - 2.4.5 Breakdowns . . . 15

3 Experiments and results **17**
- 3.1 Introduction . . . 17
- 3.2 First experiment : A robust sinus wave generator . . . 17
  - 3.2.1 Performance measurement . . . 17
  - 3.2.2 Simulation Results . . . 19
- 3.3 Second experiment : A robust square wave generator . . . 47
  - 3.3.1 Performance measurement . . . 47
  - 3.3.2 Results . . . 47
- 3.4 Concluding words . . . 52
  - 3.4.1 About the project . . . 52

3.4.2 Future investigations 52
3.4.3 Acknowledgments 53

Abstract

This report is an exploration of the robustness of populations of coupled amplitude controlled phase oscillators (ACPO). This kind of system presents interesting properties for robustness against perturbations. To show this robustness, I propose a model of perturbations like exposition to heat, local breakdowns or aging. Then I present results, obtained with the numerical simulation of the behavior of, populations of oscillators under this model of perturbations, and I propose two applications: a robust sinus wave generator and a robust square wave generator. The results are discussed in terms of profit or loss of the system's robustness. This work has been done in the frame of the Master Thesis in Computer Sciences at Swiss Federal Institute of Technology.

Chapter 1

Inspiration of the project

1.1 Introduction

The goal of this work is to explore a manner of building reliable structures, with unreliable interacting elements. In other words, inspired by the natural phenomenon of synchronization, the goal is to make a system able to resists to any perturbations like exposition to heat, local breakdowns or aging.

In order to feel from where robustness basically can come, I summarize this in one first proposal. *To be robust, a system must have no single points of failure*, otherwise, only one unfortunately breakdown of one of those single points of failure can lead to a general breakdown of the system. Similarly, a perturbation on a single point of failure will have more effects, than this same perturbation on another element. So the effect of this perturbation is not maximally attenuated by the system. This first proposal can be rewritten as : *To be robust, a system must have an output produced with the equal contribution of all elements of the system*, a non equal contribution would imply that there are elements more essential than others, and more essential elements are also single points of failure.

Actual robust systems like controllers or embedded systems, follow this proposal by duplication of the part of the system (redundancy) and complex mutual interactions (check, copy, ...). This was the only existing approach to make robust systems, before scientists thought about "How does nature proceed to be robust ?". This led them to consider some natural robust systems. Many examples exist, the most common, but very complex one, is life. Animals and plants can survive even if they are injured or ill, they do not die (stop working) immediately, like an artificial system (electrical system, computer,...). Another example of robustness in natural system is the sea waves, the sea waves are robust to sun or hot currents (i.e. any effects of heat), to boats or shallow waters, and even little islands (local perturbations).

The study of self-organization in real world complex systems has led to a global theory called synchronization [1]. Synchronization can be understood as a self-organization in natural systems, this particular behavior emerges from this self-organization. When a system is synchronized, all elements taking part of the system have similar evolution (they have a global behavior), which provides a kind of coherence. I want to demonstrate the potentiality of robustness, as one of the cues of this ensemble's coherence provided by synchronization. A lot of works on synchronization study this phenomenon with non-linear oscillators. One of the most simple systems able to synchronize is two coupled non-linear oscillators (cf. sec. 1.3.1), but synchronization can also appear in a population of oscillators. In order to link this work to the actual scientific publications, I use next section. All publication presented is part of the inspiration of this project.

1.2 Related works

As I said in the previous section, there are a lot of publications on synchronization using non-linear oscillators. Many of them use coupled non-linear oscillators and their dynamics to describe how biological components interact to produce for example: the spindle sleep rhythm [9] or circadian rhythm [10]. These works are more biological research than engineering, the comprehension of the phenomena is more important than finding an application, they simulate a system similar to real systems. For example, to simulate the circadian rhythm, H. Kunz and P. Ackermann use up to 10000 oscillators. To perform the numerical simulation of as many oscillators, they use a supercomputer (7 DEC Alpha processors). This work show the potential of coupled oscillators and discuss the tolerance to thermal noise.

M.Rosenblum and A.Pikovsky [2], the authors of one of the most complete books on synchronization (and also the most important reference of my work) [1], propose a technique to control coherent collective oscillations in ensembles of globally coupled oscillators. They demonstrate that a time delayed feedback in the mean field (cf. sec. 2.2.3) can, depending on the parameters, enhance or suppress the self-synchronization in the population. Once more, they use 2000 oscillators for their simulation, which is already enough long to simulate for a big set of parameters.

A.Ijspeert and J.Buchli [7, 8, 3], form the BIRG [1] at Swiss Federal Institute of Technology, were interested in locomotion control applied to Robotics. They have found a method to predict phase relationship between coupled phase oscillators, they also discuss robustness against noise and local perturbations. For their simulation they use a very small population composed of only four oscillators and disposed on a ring. My work extends their work by studying behavior on larger populations of oscillators (about 50) with different dispositions (one or two dimensions).

[1]Biological Inspired Robotics Group

The last paper I want to present has been written by S.H. Strogatz [2] and I. Stewart [6]. This paper, written in 1993 (more or less the beginning of synchronization's study), explain the basic dynamics of two coupled oscillators and is very abundant of biological examples. For example, he gives a very simple biological example of synchronization : the bipedal locomotion. This paper demonstrates the importance of the phenomena of synchronization in biology.

1.3 Synchronization

This phenomenon has been discovered by C.Huygens [3], when he was looking at two clocks (which he had manufactured himself) on a common support, he remarked that after a certain time both clocks started to run simultaneously. Their oscillations coincided perfectly, they were synchronized. Even if he perturbed the system, after a while readjusting their oscillations, the two clocks synchronized again. To understand what happened in the case of the two clocks, it is important to notice the weak bound that exists between the two clocks : the common support. Actually, it is through this common support, that the two clocks achieve synchronization by mutual entertainment.

The comprehension of synchronization has been extended and the phenomena is now well-described [1]. I use the forwarding section to describe more precisely what is an oscillator and how synchronization can appear with oscillators and how it will contribute to robustness of the system.

1.3.1 Non-linear oscillators

This short introduction on non-linear oscillators is taken from [3].

To understand non-linear oscillators and their behavior under perturbations, it is essential to introduce the notion of a perturbed non-linear dynamical system :

$$\dot{q} = F(q) + p \tag{1.1}$$

where q is a vector of state variables and p a perturbation vector. If the system is not perturbed ($p = 0$) and converge to a periodic solution, it is called an oscillator and the set of q on which it continues to evolve is called the limit cycle. As described in [1], all oscillators can be transformed into phase (θ) and radius (r) coordinate system :

$$\dot{\theta} = \omega_0 + p_\theta \tag{1.2}$$

$$\dot{r} = F_r(r, \theta) + p_r \tag{1.3}$$

[2]S.H. Stogatz, is well-known for his reference's book on non-linear dynamical system and chaos [5].

[3]C.Huygens,(1629-1695), the famous Dutch mathematician

where ω_0 is the natural frequency of the (unperturbed) oscillator, F_r is the dynamical system describing the evolution of r, p_θ is the component of the perturbation acting on the phase and p_r is the component of the perturbation acting in direction of the radius.

Perturbations on a stable limit cycle have different effects on the phase depending on the p_θ and p_r components. The p_θ will modify the phase, since the phase is marginally stable [1]. On the other hand, the p_r component, i.e. in the direction of the radius, will be damped out and will have little effect on the phase.

Oscillators on a stable limit cycle are *robust to perturbations in direction of the radius.* In fact, this limit cycle being an attractor (system converge to this solution), provides an other form of robustness : *robustness to initial conditions.* Actually the behavior of an oscillator is practically independent from the initial conditions..

When two oscillators (F_1, F_2 with corresponding state vectors $\mathsf{q}_1, \mathsf{q}_2$) are coupled together ($p_{\theta,2} = f(q_1)$), several types of dynamics can result including chaos and phase-locking (synchronization). But in this work, only the synchronized regimes seem to be interesting for robustness. The following section will explain how two coupled oscillators synchronize.

1.3.2 Synchronization of two coupled oscillators

Consider a system of two oscillators (F_1, F_2 with corresponding state vectors $\mathsf{q}_1, \mathsf{q}_2$) mutually coupled.

$$\dot{q_1} = F_1(q_1) + \lambda p_{c12}(q_2) \tag{1.4}$$

$$\dot{q_2} = F_2(q_2) + \lambda p_{c21}(q_1) \tag{1.5}$$

where $q_1 = [x_1, y_1], q_2 = [x_2, y_2]$, p_{c12}, p_{c21} are the interactions between the oscillators and λ is the coupling constant. When λ vanishes, each oscillators are independent and each oscillator has a stable limit cycle with constant natural frequencies $\omega_{1,2}$. In general, the frequencies $\omega_{1,2}$ are incommensurate, therefore the motion of the uncoupled oscillators is quasi-periodic (not-synchronized). But when the oscillators are coupled ($\lambda > 0$), the interactions resulting can lead to two states : frequency and phase locking.

`frequency locking` : The main point in a bidirectional interaction is that the frequencies of both oscillators change. Let me denote the frequencies of autonomous systems (called natural frequency) as ω_1 and ω_2, and let $\omega_1 < \omega_2$. The observed frequencies of interacting oscillators denote as $\Omega_{1,2}$. Then, if the coupling is sufficiently strong, frequency locking appears as the mutual adjustment of frequencies, so that $\Omega_1 = \Omega_2 = \Omega$, where typically $\omega_1 < \Omega < \omega_2$.

`phase locking` : phase locking implies a certain relation between the phases that depends not only on the frequency detuning (i.e. $|\omega_1 - \omega_2| < c$) and coupling strength, but also on the way in which the oscillators are interacting.

Consider two nearly identical, symmetrically coupled oscillators. The interaction depends in some way on the two phases, and the two simplest case are when coupling either brings the phases together, or moves them apart. Clearly, the phase-attractive interaction leads to in-phase synchronization, whereas the phase-repulsive one results in anti-phase (out-of-phase) synchronization. Moreover detuning makes the phase difference not exactly zero (not exactly π).

`high-order synchronization` : Generally, when the frequencies of the uncoupled system obey the relation $n\omega_1 = m\omega_2 + \delta$ (where δ is sufficiently small), synchronization of order $n : m$ arises for sufficiently strong coupling. The frequencies of interacting systems become locked, $n\Omega_1 \approx m\Omega_2$, and the phases are also related. We can now consider phase locking as synchronization $1 : 1$.

In this project, I am interested first of all in in-phase synchronization (sinus wave generator) and in a second time by high-order synchronization (square-wave generator). Let's see how extending this to population of oscillators.

1.3.3 Synchronization in a population of oscillators

Now we study synchronization phenomena in large ensembles of oscillators, where each element interacts with all others (globally coupled). There are many examples of synchronization in populations of oscillators in nature. The first one is "fireflies simultaneous flashing". It has been observed that, a swarm of this kind of firefly, can start flashing simultaneously. All insects in the swarm is influenced by the light field that is created by the whole population. An other example is "synchronous applause in large audience". Everybody has remarked the audience's capacity to applause synchronously. Indeed each firefly is influenced by the light field. Similarly, each applauding person hears the sound that is produced by all other people in the hall.

Synchronization can also appear in population of oscillators. The most important parameter for that is the coupling. The coupling is defined by its strength and its range. The simplest coupling manner, are the globally (all-to-all) phase-attractive (cf. sec.1.3.2) coupled population of oscillators. With this kind of coupling, the whole population synchronizes $(1 : 1)$ for a sufficiently small distribution of initial oscillator's frequencies ω_i. When the variance of the distribution of initial oscillator's frequencies increases, the oscillators begin to synchronize in several clusters. All oscillators in a cluster have the same phase, and there is a constant phase shift between different clusters.

For this work I have also tested neighborhood coupling, i.e. each oscillator is coupled with all other in a circular neighborhood of a fixed radius (the distance of coupling). This kind of coupling is cheaper in terms of number of coupling and also ensures synchronization (cf. sec.2.2.2).

For my work I use two different notions of synchronization : For the sinus wave generator I use in-phase synchronization in neighborhood coupled population of oscillators. And for the square wave generator I use high-order synchronization in neighborhood coupled "frequency-clustered" population of oscillators (cf. sec.3.3).

Now we image better how synchronization appears in coupled population of oscillators. We also see how synchronization will bring robustness to the system. Actually mutual phase-attractive interaction due to coupling will act on the system, like a shepherd with his sheep, it keeps the population coherent (gathered). This mutual phase-attraction act like this shepherd on the population of oscillators, one oscillators has a behavior depending on all coupled oscillators so all effects on it will have effects on all coupled oscillators and vice-versa. In order to measure the level of synchronization, an order parameter, called mean field (cf. sec.2.2.3), is introduce. The mean field contains all information of the behavior of the subsystems. Actually the mean field represents all informations known by the shepherd.

A system based on a population of oscillators presents a real advantage in comparison of a single oscillator system in term of robustness against oscillator breakdowns. Actually if one single oscillator is broken, the single oscillator system is whole broken but not the system based on the population.

Chapter 2

General description of the population of oscillators

2.1 Introduction

In this chapter, the description of the system (population of oscillators) (cf. sec. 2.2) and the perturbations (cf. sec. 2.4) used to in this work are presented. The software developed to simulate this population of oscillators and the perturbations is described in section 2.3.

2.2 Oscillators, coupling and output

The basic components of my system are oscillators. In order to simplify the domain of research I focus only on populations of oscillators with identical structure. All oscillators have the same behavior, if they are set to the same parameters (i.e ω_0,r,g in the case of ACPO).

For the oscillators, I have to choose between simplicity, to understand better what happens; and constructibility, to think about an easier material implementation. This choice of oscillator's type is developed in section 2.2.1, where I introduce more precisely the two different kinds of oscillators I used for my work.

Another important point is the coupling, the most used (and described) techniques of coupling are globally coupled or neighborhood coupled (cf. sec. 1.3.3), but coupling is also dependent on the pattern and dimension of the population. I describe the models of coupling in section 2.2.2.

The last part of the conception of the system is the output of the system. Actually I said into introduction that robust systems must have an output, function of all elements and not to complicated. This point is developed in section 2.2.3.

To simulate the system I have implemented software that makes a numerical integration of the non-linear differential equation which describes my system. This software allows users to create and simulate (numerically) a population of oscillators by specifying few parameters (cf. table 2.1).

2.2.1 Oscillators : amplitude controlled phase oscillator vs Van der Pol's oscillator

The simplest oscillator is the phase oscillator (cf. eq. 2.1), it is a mathematical oscillator, in the sense that it can hardly be constructed. Its limit cycle is a harmonic circle for every values of parameters and so the variables of the oscillators describes (in the case of an unperturbed system) a perfect sinus wave. In fact, the phase oscillator can be extended to the Amplitude Phase Controlled Oscillator (ACPO) and keeps the properties of the phase oscillators [3]. In my work I first use ACPO because it was easier to work with perfect sinus (for example : the Fourier frequency analysis of a perfect sinus is a single peak), almost because with this oscillator I could be sure of having the same behavior (i.e limit cycle and form of oscillation) for every set of parameters. This was very important for my work to have *only wanted perturbations*, to see the real effect of the perturbations I want to study (cf. sec. 2.4). The problem is that all results are to be study, keeping in mind that the system is very hard to manufacture. But I expect having a comparable behavior with the other types of oscillators (with adequate parameters).

The ACPO is described by these following equations :

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{r} &= -g(r - r_0) \end{aligned} \tag{2.1}$$

where r_0 set the amplitude of oscillation, g set the damping of oscillation and ω is the intrinsic frequency of the oscillator. ACPO is easily configurable because, it three parameters are independent (this is not the case for VDP oscillator).

Equation 2.1 can be transformed in a Cartesian coordinated system :

$$\begin{aligned} \dot{x} &= g(\tfrac{r_0}{\sqrt{x^2+y^2}} - 1)x - y\omega \\ \dot{y} &= g(\tfrac{r_0}{\sqrt{x^2+y^2}} - 1)y + x\omega \end{aligned} \tag{2.2}$$

In equations 1.4,1.5, F_1 and F_2 can be replaced by equation 2.2 to find the system of ODE, the program has to integrate, to simulate the behavior of the coupled population of oscillators.

The second kind of oscillator I use, is called Van der Pol (VDP) oscillator (cf. eq. 2.3). It is part of an other class of oscillators called relaxation oscillators, because the form of oscillation is no more a simple sine wave; rather, it resembles a sequence of pulses [1]. In fact, for small μ this oscillator is quasilinear (i.e limit cycle is near a circle), while for large μ it is of a relaxation type (i.e. the limit cycle is no more a circle). Equation of the

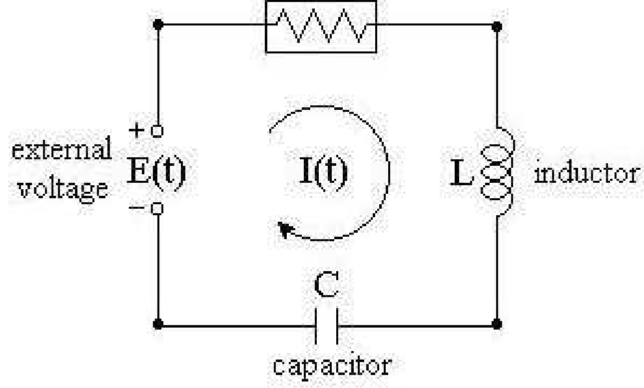


Figure 2.1: This is the physical implementation of the Van der Pol oscillator

VDP oscillator :

$$\ddot{x} = 2\mu\dot{x}(1.0 - \beta x^2) - \omega^2 x \tag{2.3}$$

Where μ sets the general form of oscillations (for $\mu = 0.1$ the limit cycle is close to a circle), β sets the damping of oscillation and ω is To be integrated by my program, equation 2.3 must be separated in a system of two first order ODE.

$$\dot{p} = [\dot{x}, \dot{y}]^T = [y, 2\mu y(1.0 - \beta x^2) - \omega^2 x]^T \tag{2.4}$$

Contrarily to the ACPO, the Van der Pol oscillator is constructible with a simple electric circuit composed by a capacitor, a neon tube, a resistance and, of course, a battery. This is the main interest of this oscillator : with it we can think about a very simple physical implementation (cf. fig. 2.1).

2.2.2 System's coupling : neighborhood coupled

The most common coupling is `globally coupled`, this coupling is simple but have a disadvantage : a globally coupled population of N oscillators have a $O(N^N)$ number of coupling. In practice it would be better to have less interaction than possible, because it would be easier implementable. For that I use a $n$eighborhood coupled population of oscillators. In this population, all oscillators are coupled with all other oscillators in a neighborhood. The distance of coupling d_{coupl} (size of neighborhood), having for unit one oscillator, is one of the most important parameter in my system. With this model of coupling I can vary from next neighbor coupling ($d_{coupl} = 1$) to globally coupled.

As developed in 1.3.2, coupling represents interactions, these interactions can be either mutual or unidirectional, phase-attractive or repulsive. But in my work, I always use mutual and phase-attractive interaction because, in all experiments, I need to achieve in-phase synchronization.

So for an oscillator i, the coupling corresponds to :

$$\dot{q}_i = F(q_i) + \lambda \sum_{j=1}^{n} p_{cij}(q_j) \tag{2.5}$$

where $p_{cij} > 0$, n is the number of coupled oscillators (dependent of the distance of coupling d_{coupl} and λ the coupling force.

In order to have comparable coupling between two different populations with different size, the sum of interactions of an oscillator is about the amplitude of the oscillator, so is normalized by the number of coupled oscillators, so :

$$\sum_{1}^{n} p_{cij}^{norm}(q_j) < A \tag{2.6}$$

where $p_{cij}^{norm} = \frac{p_{cij}}{N}$ (N = size of the population) and A is the amplitude of oscillations.

Neighborhood coupling is dependent on the geographical disposition of the oscillators. Actually, the number of coupled oscillator is dependent on the dimension and the form of the population : in a line of oscillator, a coupling distance of one will couple the two nearest neighbors, instead of the four (or eight if we take the diagonal) if the population is in two dimensions. Moreover, if we transform the disposition of oscillators from line to closed-line (annulus) or from plane to torus, it leads also to a different number of coupling. So I also study the impact of the population dimension (i.e. one (line) or two (plane) but always closed (annulus or torus).

2.2.3 System's output : mean field

The question was to find a good output for my system, easy to calculate and robust. One natural idea was the mean field. The mean field (cf. eq. 2.7) M is the average of all "signal" s_i in the population of N oscillators.

$$M = \frac{1}{N} \sum_{i=1}^{N} s_i(t) \tag{2.7}$$

It's an easy way to measure the output of the system and takes into account all informations of the system. Moreover, in some particular example, it has a realistic meaning : the magnetic effect of a magnetic material can be seen has a mean field of magnetic force. The mean field can also be taken as an order parameter. In my system of coupled population of ACPO, the mean field is the average of quasi-pure sinus wave. The mean field is also a sinus wave, if the oscillators are all in-phase synchronized. But if the synchronization is not complete (extended to the whole population), there is a progressive degradation of the mean field that can leads to an average of amplitude of the mean field is equal to zero in the case of the oscillators phases are uniformly distributed (cf. fig. 2.2

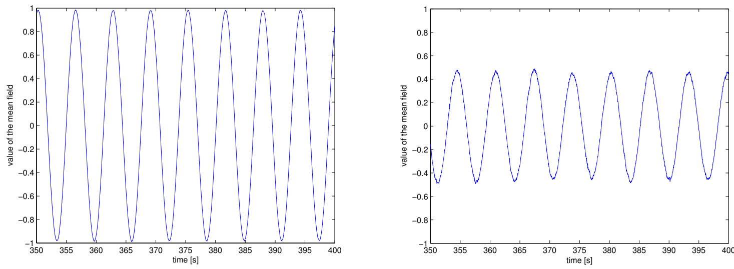


Figure 2.2: typical mean field : left) complete synchronization right) 50% of synchronization

2.3 Implementation and numerical simulations

All simulations are performed with the software developed in the frame of this work. This software has been implemented in C and uses the Gnu Scientific Library (GSL). GSL is a good and reliable scientific library. Differential equations were integrated using a fourth order Runge-Kutta algorithm with an adaptive step-size. One must remember that an adaptive step-size means an uncontrolled (unpredictable) runtime of the simulations, but also produce more precise results. Nevertheless, in order to perform the data processing on a equidistant sampling, only the data corresponding to a fixed step-size are stored.

For reasons of performance, data analysis is made straight in the software with GSL, but additional analysis has been made with Matlab. GSL do not propose the Hilbert Transform (used in first experiment), algorithms used in performance measurements, so I implemented it and verified with Matlab.

In all further simulations, I use a population of about 50 identical ACPO distributed along a annulus (1D) or a torus (2D), and I study the behavior during 500 [sec]. With 50 oscillators the run-time is acceptable ($< 2min$)for any perturbations but for more oscillators (> 100) run-time becomes too big ($\approx 10min$). Taking into account the adaptive step-size, and the different perturbations and the run-time became really important ($\approx 15min$). All experiment done in this work are study of parameters, which implies a large set of value to test. So one experiment can lead, in the worst, case to simulation's duration of 2 or 3 days. As my work consists in making a lot of experiments, about a hundred (bad and good are counted), 2 or 3 days for one only is too long. In a further time, one should simulate such complex system (population of oscillators) with super-calculators (simultaneous multi-processing), to keep a relative flexibility of work.

In the case of an unperturbed system, all oscillators have the same initial conditions ($[x_0, y_0]$) and parameters (ω, g, r_0). The coupling used is neigh-

system parameter	symbol	values	description
coupling force	λ	$[10^{-6}, 50]$	strength of the interactions
coupling distance	d_{coupl}	$[0, 1]$	size of neighborhood
number of oscillator	n	$[1, 250]$	$\approx$ time of simulation
dimension	–	$[1 or 2]$	annulus or torus
oscillator type	–	$[0 or 1]$	ACPO or Van der Pol

Table 2.1: Parameters of the population of oscillators

borhood coupled, described in section 2.2.2. This population of oscillator, with a sufficiently strong coupling but for any value of distance of coupling, synchronizes easily and quickly, and then the behavior of the system is very stable. By choosing right initial conditions, it is possible to start immediately with a synchronized population. I use this little trick to allow me having a stable system, already at the beginning of simulation, so that simulation time (processing time) can be shorter.

I have tested the performance of the system by studying the impact of all perturbations, presented in section 2.4, on the system modulated by five different parameters (cf. table.2.1) :

2.4 Perturbation's model

To test the robustness of the different coupled populations of oscillators, I propose a model for some types of perturbations. All these perturbations can occur in real conditions, their are all perturbations present in real world. The most basic natural sources of perturbations are heat, shocks or aging. I have extended the model of perturbations to include also imprecision of manufacture, and the variation of initial conditions, but the sources of perturbations are mainly natural and so they are common to all systems. So this model of perturbations is more or less generic.

The table 2.2 resumes all the perturbations I have simulated and their model. All perturbations are explained more precisely in next sections.

2.4.1 Thermal noise

The most common kind of perturbation is due to heat. Actually, heat introduces a strong activation of the material, what modifies the behavior of the whole system proportionally to the level of heat. In engineering this problem is well know and, the more commonly used model used for thermal noise is modeled by a reduced and centered Gaussian distributed random value η_{gauss} added to the variables of the system. To modulate the power of thermal noise, I introduce a thermal noise parameter P_{thn} between $[10^{-2}, 10^{2}]$. So each step of the numerical integration, I added $P_{thn}\eta_{gauss}$ to the variables of the system. Notice that, as my numerical integration step is adaptive, I

perturbation	parameter	source	model
thermal noise	P_{thn}	exposition to sun, or to another heat source	increase noise on all system interactions
fault of fabrication	P_{fd}	imprecision of manufactured material	deviation of the fabrication parameters
component aging	P_{cpa}	degradation of the material due to age	variation of the fabrication parameters
variation of initial conditions	P_{ic}		variation of the inittial conditions
breakdowns	R_{bkd} f_{bkd}	shocks or bad use of the system	breakdown on oscillators or coupling

Table 2.2: This table summarizes all the perturbations implemented for this work

don't know exactly how much noise has been added during a simulation. In fact it adds another little random component to thermal noise. An other model for thermal noise is a uniformly distributed, in [-1,1], random value, but this "uniform" noise is less perturbing than a "Gaussian" noise (i.e. "uniform" noise is bounded), and I know that oscillators are naturally robust to thermal noise. So for all simulations I use a "Gaussian" noise.

In presence of thermal noise, equation (2.1) becomes :

$$\frac{dp}{dt} = F_{acpo}(x(1 + P_{thn}\eta_{gauss}), y(1 + P_{thn}\eta_{gauss})) \tag{2.8}$$

We already know that a single oscillator is robust to thermal noise ([1]), but I expect the population of oscillators to be more robust.

2.4.2 Faults of fabrication

A material oscillator, like every manufactured products, has an intrinsic precision of fabrication, which is described by the incertitude on the fabrication values.

This can be modeled by adding a uniformly distributed random value η_{unif} to the initial parameter of the oscillators (ω, g, r_0). I uses a uniform distribution because I want bounded faults of fabrication. η_{unif} is modulated by the fabrication faults parameter P_{fd}

In fact I focus on the fault of fabrication to the study of the initial distribution of ω because it seems to be the more influential parameter, but it should be also interesting to study g or r_0. The initial distribution of

ω is one of the most important and studied parameter in a population of oscillators, as we have seen in section 1.3.3, a to much big initial distribution of ω prevent the population from synchronization.

2.4.3 Component aging

The aging is not very frequently simulated, but is ever present in nature and has a strong effects on material (progressive degradation, breakdowns, bad contact, ...). To model the degradation of the system due to age, time-dependent parameters are introduced, so for example: ω becomes $\omega(t)$. The aging corresponds to modify the parameter of the system by adding a centered and reduced Gaussian distributed random value η, controlled by a component aging parameter P_{cpa}, in function of time, and keep this modified parameter for the next step of the numerical integration random walk. Evolution of a parameter, taking account of aging is described by following equation:

$$\omega_{n+1} = \omega_n + \eta(\sigma) \tag{2.9}$$

n+1 does not corresponds to the next adaptive step of the numerical integration, but to a fixed time step of 0.1 [sec].

2.4.4 Variation of initial conditions

Robustness to variation of initial conditions is an important characteristic of a system. Indeed, the robustness to variation of initial conditions is very appreciated as system's properties, it let the initialization of the system be free. Robustness to initial condition determine also if the system is chaotic or not, actually chaotic are very few robust to variation of initial conditions. To model the variation of initial conditions, a uniformly distributed random value η_{unif} is added to initial conditions.

2.4.5 Breakdowns

The breakdowns are probably the most problematic kind of perturbations because, in an electronic system in general, a breakdown implies, the more often, the complete crash of the system. I my system I expect a better behavior. I propose two kinds of breakdowns, on oscillators and on coupling. When an oscillator breaks, one can imagine almost two kinds of behaviors : the oscillator keeps its current value (constant value) or the oscillator gives a totally random value, it become a noise generator. I have modeled a coupling breakdown by the suppression of a randomly chosen coupling in my population. All breakdowns are controlled by two parameters, the breakdown frequency f_{bkd} and the maximal percentage of broken oscillators R_{bkd}. f_{bkd} is bounded by [0,1]. A frequency equal to one implies a breakdown at each fixed steps (0.1 sec) of the numerical simulation. R_{bkd}, the percentage of breakdown set the maximal number of oscillators potentially breakable in the population. In practice, I set the frequency of breakdown in a way to have all breakdowns in, approximatively, the first 50 sec of the simulation run, and I do vary only the broken oscillator's percentage. This allows to have a simple control on the number of oscillators broken during the simulation.

Chapter 3

Experiments and results

3.1 Introduction

All experiments presented are the results of numerical integration of a population of neighborhood coupled oscillators (ACPO). More precisely, next sections are used to describe the behavior of a coupled population of oscillators in two particular applications. The first one, a sinus wave generator and the second one, a square wave generator. The behavior of these two applications is discussed in terms of robustness to the perturbations described in section 2.4.

3.2 First experiment : A robust sinus wave generator

A single oscillator can be considered as a sinus wave generator, so naturally the simplest application we can imagine using population of oscillators is a robust sinus wave generator. In theory, all the oscillators of the population, under phase-attractive coupling, synchronize in-phase and produce a very robust mean field equal to a perfect sinus.

3.2.1 Performance measurement

In this section, it is explained how the performance of the system is measured. In fact, as we have just seen in 2.2.3, the mean field of a single population of oscillators is a perfect sinus wave when the synchronization is complete (all oscillators in-phase synchronized). So to know if a population is synchronized and in which proportion, a simple manner is to compare the mean field with a perfect sinus wave, using the Hilbert transform, as explained in next section.

Level of synchronization : average and standard deviation of the mean field

I use the fact that the Hilbert transform (HT) of a sinus wave is a circle in complex plane. So if the norm of the HT of a signal is constant, this signal is a sinus. Moreover, with a normalized mean field, we have a good approximation of the instantaneous percentage of synchronization $\sigma(t)$ of the system. We can write :

$$\sigma(t) = norm(HT(mf))(t) \tag{3.1}$$

Then I take the average and standard deviation on time of the instantaneous state of synchronization. This gives me a good idea of the rate of synchronization along time. For example, if the average is equal to one and the standard deviation small enough, one can be sure that, the mean field is a perfect sinus wave and all oscillators are in-phase synchronized during all the time of simulation (fig. 2.2 a)).

Quality of the generator : frequency stability of the principal component and level of noise of the mean field

The goal is to obtain a robust sinus wave generator, able to keep providing a sinus wave with the right frequency (minimum requirement) and with an acceptable level of noise (the signal must be easily isolated), in presence of perturbations. An typical acceptable level of noise is a signal to noise ratio of at least a hundred [12]. From the Fourier transform (FT) of the mean field, we can easily extract the frequency corresponding to the sinus generated and the level of noise.

The frequency of the principal component of the mean field and it's ratio with noise are the two basic tests of quality apply to the system. The frequency stability is important but don't gives a sufficiently indications on the quality of the signal, this is why the measure of the signal to noise ratio was added (fig. 2.2).

system parameter	symbol	default value
coupling strength	λ	0.05
coupling distance	d_{coupl}	0.5
number of oscillator	n	49
dimension	–	1
oscillator type	–	ACPO

Table 3.1: Table to summarize the default value of the parameters. For each following experiments, if the parameter's value in not specified, it was set to its default value.

3.2.2 Simulation Results

The simulation results of the sinus wave generator are presented in order to regroup all effects of each perturbations according to the maximum of different system's parameters. In order to have a mean field, when the whole population is synchronized, of amplitude equal to one and a principal frequency of 1 [Hz]. I set the oscillators with an intrinsic frequency of 1 [Hz] and an amplitude equal to one, and for the ACPO, $r_0 = 1$ and $g = 10$. Nevertheless in the simulations, the principal frequency is never exactly equal to 1, this is due to the precision of the numerical Fourier Transform, dependent on the sample's size and the duration of the simulation. Before all, I make a little parenthesis, to verify that the property of robustness to initial conditions described in section 1.3.1 for one oscillator, is transmitted to population of oscillators. In the following sections all simulations last 500[s] and the default parameters summarize in table 3.1.

Variation of initial conditions

As expected the population of oscillators is very robust to initial conditions. This experiment shows if the system is able to synchronize during the simulation time (500[s]), and so gives an idea of the speed of synchronization (cf. fig. 3.2). Fig. 3.1 shows that the population synchronizes for random initial conditions between [-1,1] for a weak coupling ($\lambda < 1 * 10^{-2}$), and then if the coupling increases reaching $\lambda \approx 1 * 10^{-1}$ the system is totally synchronized for initial values in [-40,40]. This experiment demonstrates also that the stronger the coupling is, the faster the synchronization achieves. For all experiments that follows, initial conditions are no more discussed, because the initial condition are set in order to have a system synchronized since the beginning of the simulation.

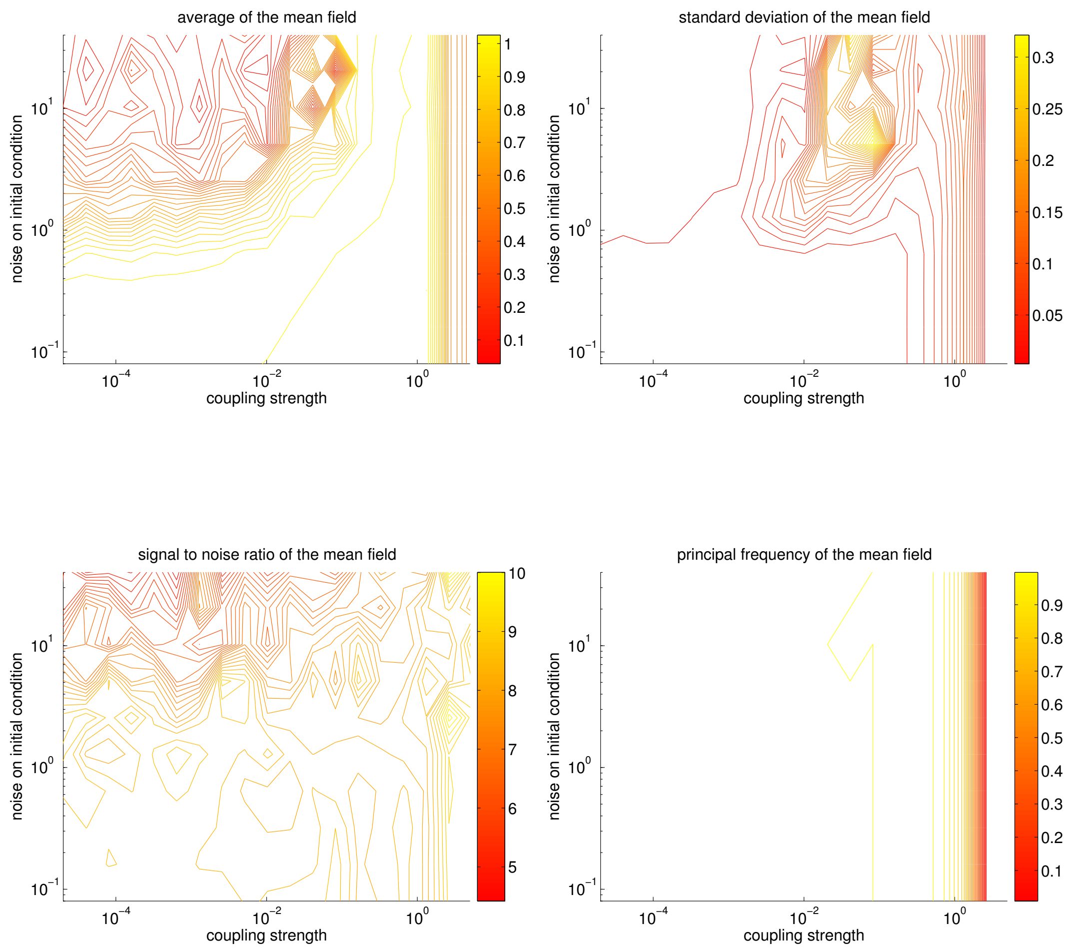
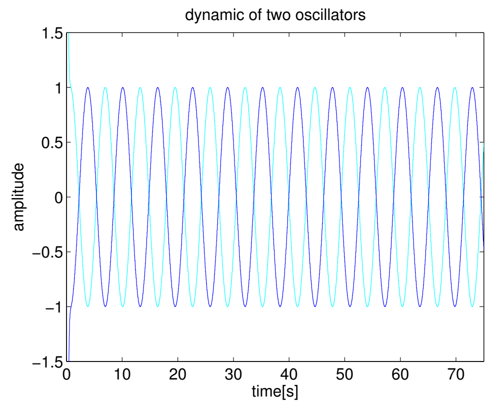


Figure 3.1: Impact of the coupling strength in presence of variation on initial condition. For a too large distribution of initial conditions, the system is not able to synchronize during the time of simulation (500[s]), so the average of the mean field decreases (cf. fig. 3.2). A stronger coupling increases the robustness of the system, the average of the mean field stay equal to one for a larger distribution of initial conditions. Nevertheless the system is completely robust to initial conditions between for any strength of coupling, because the frequency is always stable until the strength of coupling exceeds $\lambda \approx 1 * 10^{-1}$.



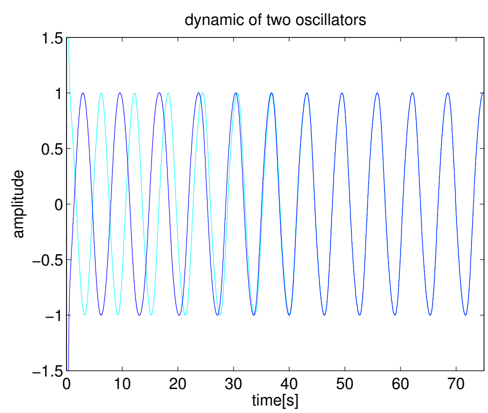


Figure 3.2: Impact of the coupling strength in presence of variation on initial condition. These two plots represent one of the variable of two particular oscillators in the population. For a little distribution of initial conditions (bottom plot) the oscillators achieves synchronization in less than 50[s] and for the same distribution (top plot) the oscillators are still not synchronized after 70[s]

Impact of the thermal noise

Fig. 3.3 and fig. 3.4 shows the general behavior of the sinus wave generator in presence of thermal noise, for different values of coupling strength. The increase of couping strength do not increase a lot the system's marge to thermal noise ($P_{thn}^{max} = 0.15 -> 0.2$), but it has others effects. Actually increasing the coupling strength, makes the region of high standard deviation of the mean field getting thinner . For a strong coupling $\lambda \in [10^{-1}; 1]$, the standard deviation do not increase a lot during the transition phase (Fig. 3.4 arrow on std plot). Focalizing on the frequency stability and the signal to noise ratio, one can see that, even if the system is not completly synchronized and has not a minimal standard deviation, the frequency is stable with a very acceptable signal to noise ratio ($snr > 5$) for the whole region a) delemited by the dashed line. The sinus wave generator is totally robust against thermal noise, while the level of thermal noise do not exeeds $P_{thn} \approx 1$. A superior level of thermal noise ($P_{thn} > 1$) leads to a progressive degradation of the frequency of the generator. For the other experiments on thermal noise the coupling's strength is set to 0.05; it is enought to provide a good robustness to thermal noise and not to strong to keep a relative not too forced population.

Fig. 3.5 and fig. 3.6 are respectively the results of the simulation for different size of population and for different distance of coupling. The size of population and distance of coupling are importants parameters which determine the general complexity of the system, it has a lower cost in processing time and an hypothetic manufacturing would be easyer. One of the worst case, in term of complexity, would be that the best for system's robustness, is to have the maximum of oscillators and the maximum of coupling. Fortunatelly, fig. 3.5 seems to demonstrate that in the case of thermal noise after a certain number of oscillators, about thirty, the robustness do not increase a lot. In the same way, fig. 3.6 do not reveal any important impact for distance of coupling in regard to tolerance for thermal noise, only that no coupling leads, for $P_{thn} > 0.2$, to a bigger standard deviation of the mean field and a smaller signal to noise ratio, but frequency remains stable.

For this kind of perturbation, actually thermal noise, the gain in robustness is not sufficient to justify the use of a more than next neighbor coupled (d_{coupl}=1 oscil) population of oscillators. At the opposite, more than thirty oscillators, and a sufficiently strong coupling is indicated to increase the system's robustness to thermal noise.

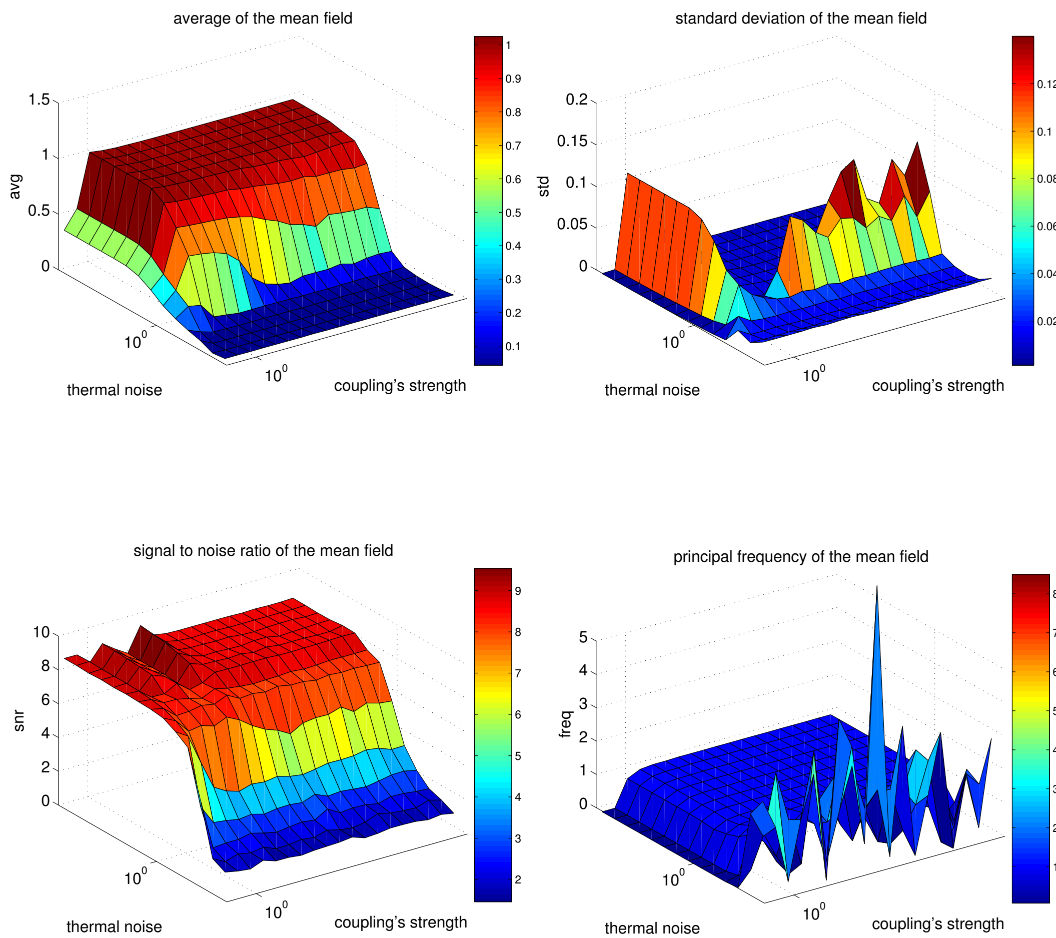


Figure 3.3: Impact of the coupling's strength $\lambda \in [10^{-4}; 10]$ in presence of thermal noise $P_{thn} \in [0.02; 10]$. All other parameters are the default's one. These four plots , describe the general behaviour of the population in presence of thermal noise; the domain of synchronization (i.e avg=1) corresponds to the domain of maximal signal to noise ratio and to the domain of stability of the frequency. Moreover the phase transition between the synchronized state(i.e avg=1) and the non-synchronized state (i.e $avg \approx 0$), corresponds to the maximum of standard deviation of the mean field. This show that we are in presence of a powerlaw for the parameter of thermal noise. One can also notice that, there is no significant increase of tolerance to thermal noise due to the increase of coupling's strength. Moreover a stronger coupling than $\lambda \approx 2$ leads to oscillation's deapth and the loss of synchronization.

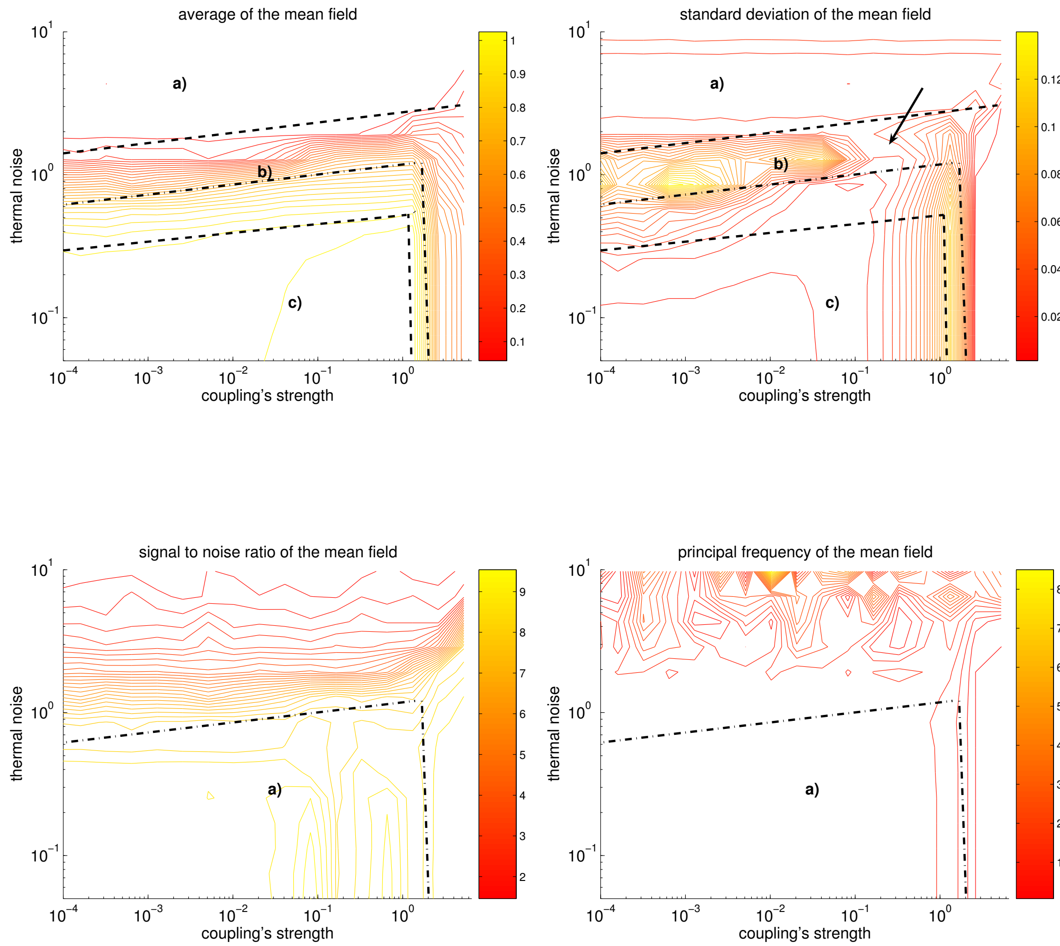


Figure 3.4: Impact of the coupling strength $\lambda \in [10^{-4}; 10]$ in presence of thermal noise $P_{thn} \in [0.02; 10]$. This is exactly the same simulation than for fig. 3.3. We can see more precisely the syncronization region (i.e domain c) closed by the dashed line), and the phase transition area (i.e domain b) closed by the two dashed lines. The dotted-dashed line corresponds, in the four plots, to 50% of synchronzation. We can see that 50% of synchronization is enougth to keep a stable frequency and an acceptable level of noise ($snr > 7.5$). The transition's region(b), corresponds to the maximal standard deviation region (powerlaw). Increasing the coupling strength has a little effet on system's marge to thermal noise, we can see that for a weak coupling strength ($\lambda \in [10^{-4}; 10^{-3}]$), the level of thermal noise corresponding to the change from synchronization's state to transition state (b) is equal to $P_{thn}^{max} \approx 0.15$, and for a strong coupling $\lambda^{max} \in [10^{-1}; 1]$ the maximal level of thermal noise $P_{thn}^{max} \approx 0.2$. (on plot of avg)

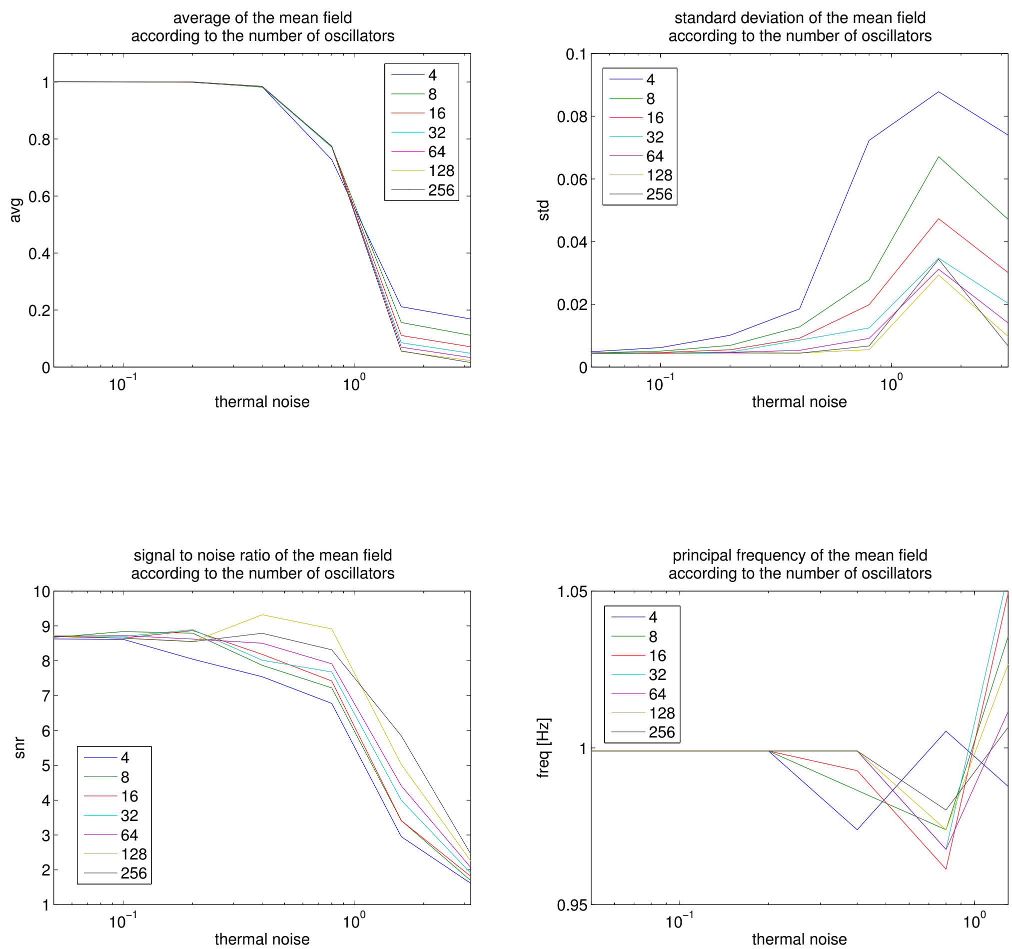


Figure 3.5: Impact of the number of oscillators $n \in [4; 256]$ in presence of thermal noise $P_{thn} \in [0.05; 2]$. The number of oscillators (i.e size of the population) influences the general behavior of the sinus wave generator : one can see that the standard deviation decreases with the number of oscillators, while the number of oscilators do not exceeds 32. For more than 32 oscillators the general behavior do not more vary in a significant manner.

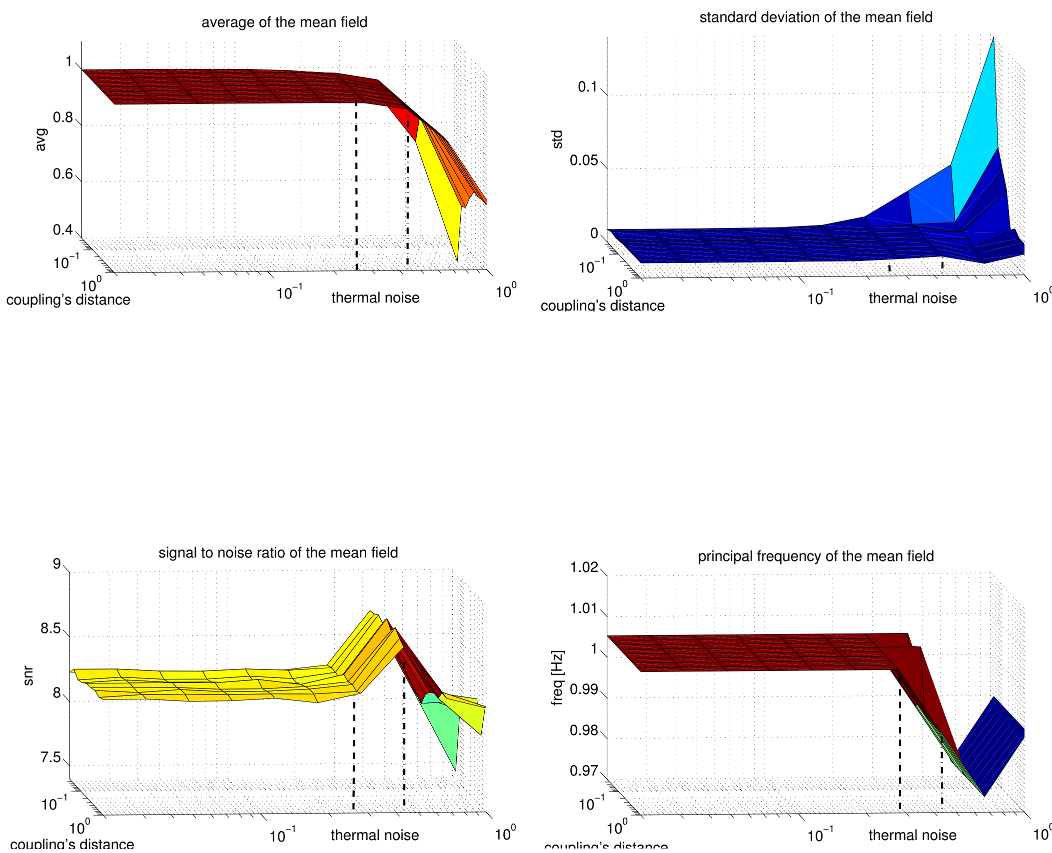


Figure 3.6: Impact of the coupling distance $d_{coupl} \in [0.02; 1]$ in presence of thermal noise thermal noise $P_{thn} \in [0.02; 1]$. No coupling is clearly less robust than any other coupling's (avg, std, snr plots) but has no impact on frequency stability. From the principal frequency plot, we can extract the system's tolerance to thermal noise: to keep a 5% stable generator P_{thn} must be < 1 and to keep a perfectly stable generator $P_{thn} < 0.2$.

Impact of fabrication's default

Fig. 3.7 and fig. 3.8 show the phase transition taking place when the level of fabrication default increases. For a relative weak coupling $\lambda = 0.01$, the maximal marge to fabrication's default is equal to $P_{fd}^{max} 7 * 10^{-3}$ (cf. fig. 3.8). This marge increases when the coupling becomes stronger; $\lambda = 0.1$ corresponds to a tolerance equal to $P_{fd}^{max} \approx 3 * 10^{-2}$. This marge increases too if the population of oscillators is in two dimensions (torus). Indeed, for $\lambda = 1 * 10^{-2}$, the max tolerance is equal to $P_{fd}^{max} = 2 * 10^{-2}$, more or less three times more robust than in one dimension (cf. fig.3.9). As in the case of thermal noise, the effects of the size of the population and distance of coupling have been simulated. Fig.3.10 demonstrates that more than four oscillators are requested to increase the system's marge to fabrication's default. But also that the degradation of the frequency is slower in this case (n=4). Indeed, for $P_{fd} = 0.1$, if n=4 frequency= $1 \pm 2\%$, n=16 frequency= $1 \pm 10\%$,and for n=256 frequency= $1 \pm 10\%$. Fig. 3.11 and fig. 3.12 show that contrary to thermal noise, the system's marge to fabrication's defaults is dependent on the distance of coupling. Actually, until $d_{coupl} \approx 1.5 * 10^{-1}$(i.e 15% of the population's size), the marge increases from $P_{fd}^{max} = 1.8 * 10^{-2}$ to $P_{fd}^{max} = 3 * 10^{-2}$. Moreover, this figure shows that a larger coupling is not necessary, because it do not increases the robustness (cf. fig.3.12). Taking account of these results, it is clear that it is possible to increase the robustness against fabrication's defaults by choosing a population in two dimension and an appropriate distance of coupling. A population in two dimensions (i.e a torus) of $n > 4$ oscillators, with a distance of coupling $d_{coupl} > 0.2$ and a maximal coupling strength λ is the more robust configuration against fabrication's defaults.

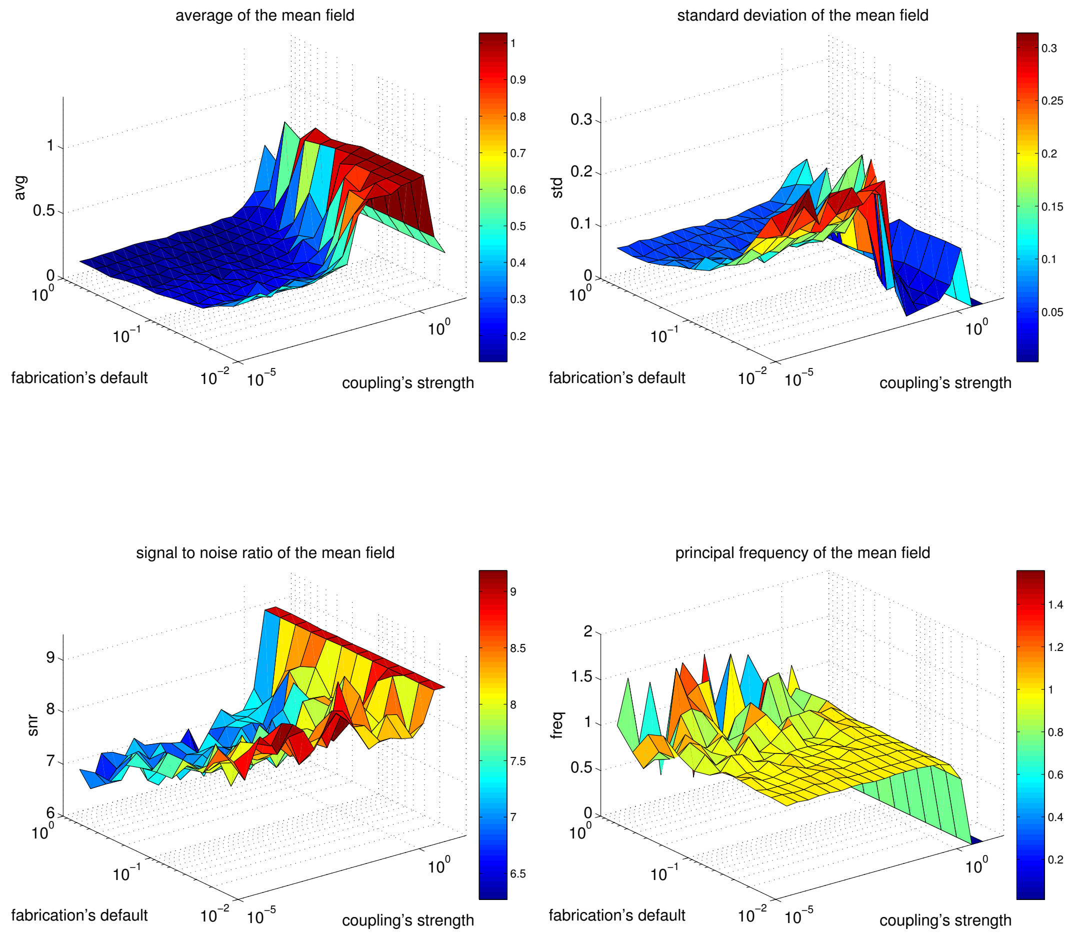


Figure 3.7: Impact of the coupling strength $\lambda \in [10^{-5}; 2]$ in presence of fabrication's default $P_{fd} \in [0.02; 1]$. These four plots, shows the phase transition between synchronized state (avg=1) and non-synchronized state (avg≈0) resulting from the increase of the level of fabrication's defaults. The system's marge to fabrication's default varies in function of the coupling strength. Indeed, one can see that for a coupling strength $\lambda < 0.01$, $P_{fd} = 1 * 10^{-2}$ is enought to loose synchronization, and for a strong coupling $\lambda = 1$ the phase transition occurs when P_{fd} exeeds $3 * 10^{-1}$. As before, a coupling strength $\lambda > 2$ leads to oscillation's deapth and to the complete degradation of the frequency.

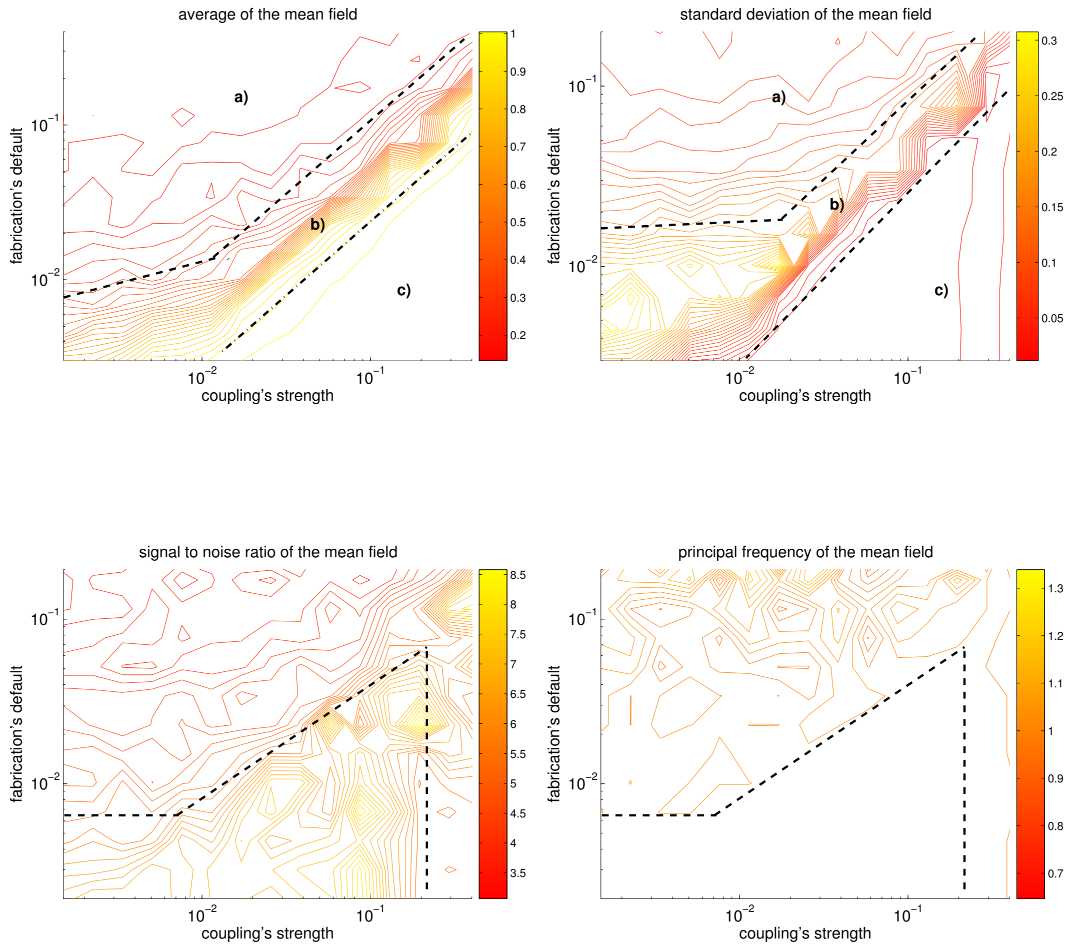


Figure 3.8: Impact of the coupling strength $\lambda \in [1.5^{-3}; 0.2]$ in presence of fabrication's default $P_{fd} \in [2 * 10^{-3}; 0.2]$. On the two upper plots, c) is the region of synchronization, b) the transition region and a) the region of non-synchronizion. The dotted-dashed line corresponds to 50% of synchronization. The transition region (b) corresponds to the region of maximal standard deviation. The coupling strength influences the system's tolerance to fabrication's default. Indeed, for a coupling strength $\lambda < 6 * 10^{-3}$, the frequency remains stable until the level of fabrication default reaches $P_{fd}^{max} = 6 * 10^{-3}$. Then if the coupling strength increases, the robustness of the system to fabrication's defaults increases too, reaching $P_{fd}^{max} = 6 * 10^{-2}$ for $\lambda \approx 0.2$. Signal to noise ratio is maximum on the whole domain of stable frequency, and is always greater than seven, the noise is so at least 10^7 times smaller than the signal.

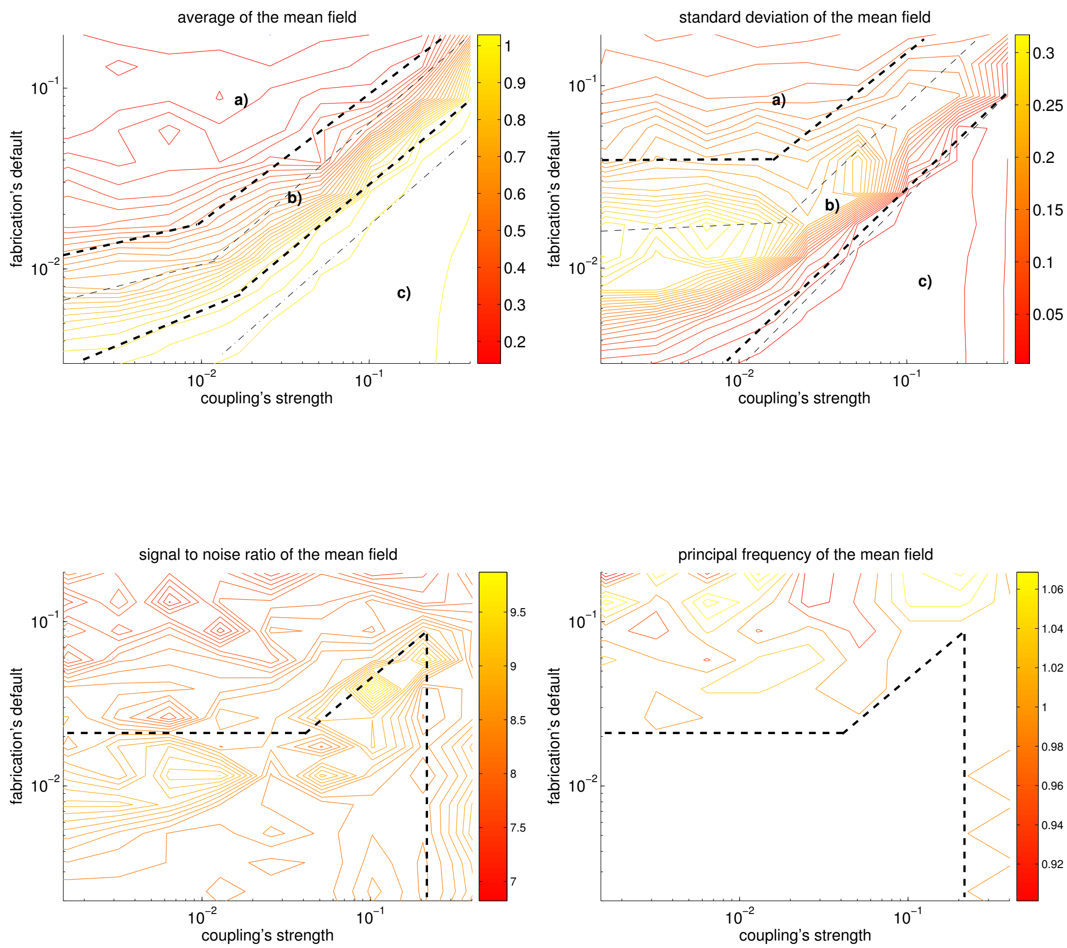


Figure 3.9: Impact of the coupling strength $\lambda \in [1.5^{-3}; 0.2]$ in presence of fabrication's default $P_{fd} \in [2*10^{-3}; 0.2]$ for a population in two dimensions. Exactly the same parameters than for fig. 3.8, only dimension changes. We can see that the system increases in robustness, passing from one to two dimension. Indeed, in two dimensions, the frequency is stable for a higher value of fabrication's default $P_{fd}^{max} = 2*10^{-2}$ for weak coupling $\lambda < 0.01$, but for stronger coupling $\lambda > 0.05$ the dimension does not make a big difference.

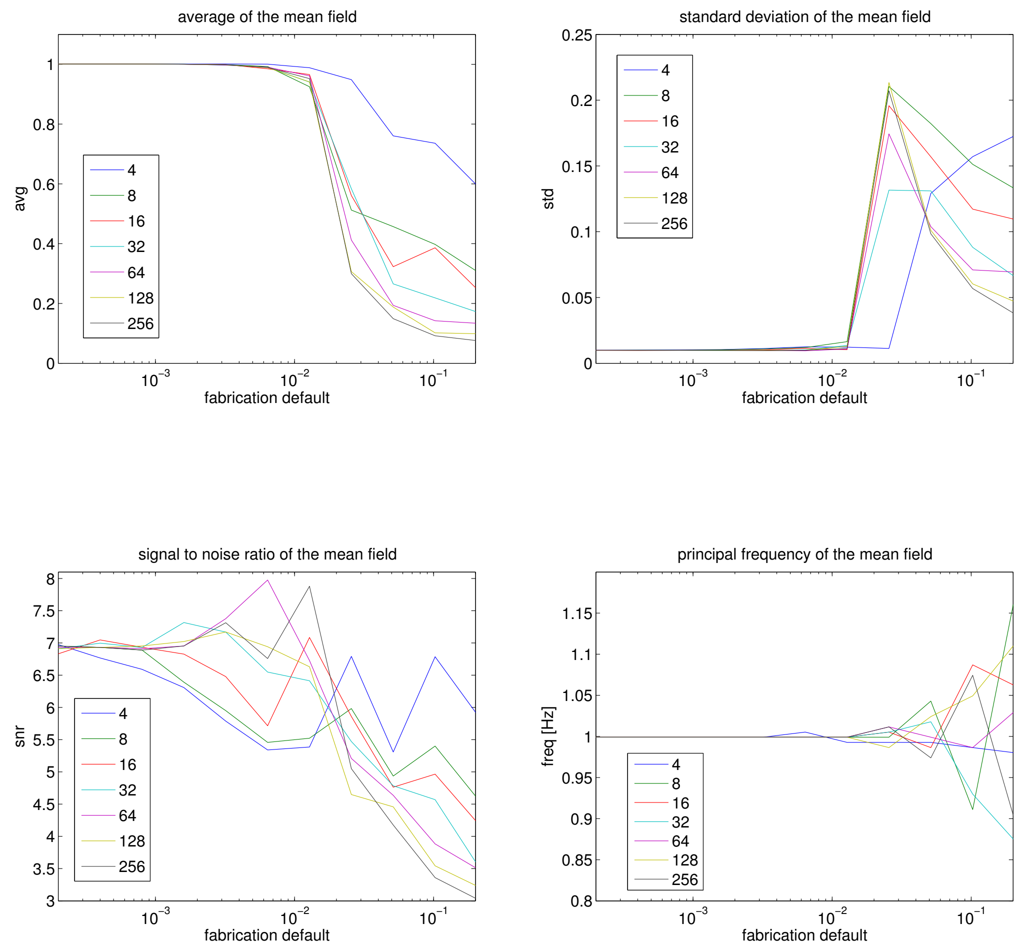


Figure 3.10: Impact of the number of oscillators $n \in [4; 256]$ in presence of fabrication's default $P_{fd} \in [2 * 10^{-3}; 0.2]$. The case with four oscillators seems to stay synchronized (avg=1, std=0) for a higher level of fabrication's default $P_{fd}^{max} \approx 2.5 * 10^{-2}$, instead of $P_{fd}^{max} \approx 1.2 * 10^{-2}$ for more oscillators. In fact in terms of signal to noise ratio and frequency stability, one can see that $n = 4$ is not always the best case. In the case n=4 the frequency is stable while $P_{fd}^{max} < 2.2 * 10^{-3}$, and for $n > 4$ $P_{fd}^{max} < 1.5 * 10^{-2}$. Which corresponds to value found in fig. 3.8 for a coupling strength $\lambda = 0.05$. Nevertheless the case n=4 seems to keep the frequency in a more acceptable range (2%) of variation, for strong fabrication's default $P_{fd}^{max} > 1 * 10^{-2}$.

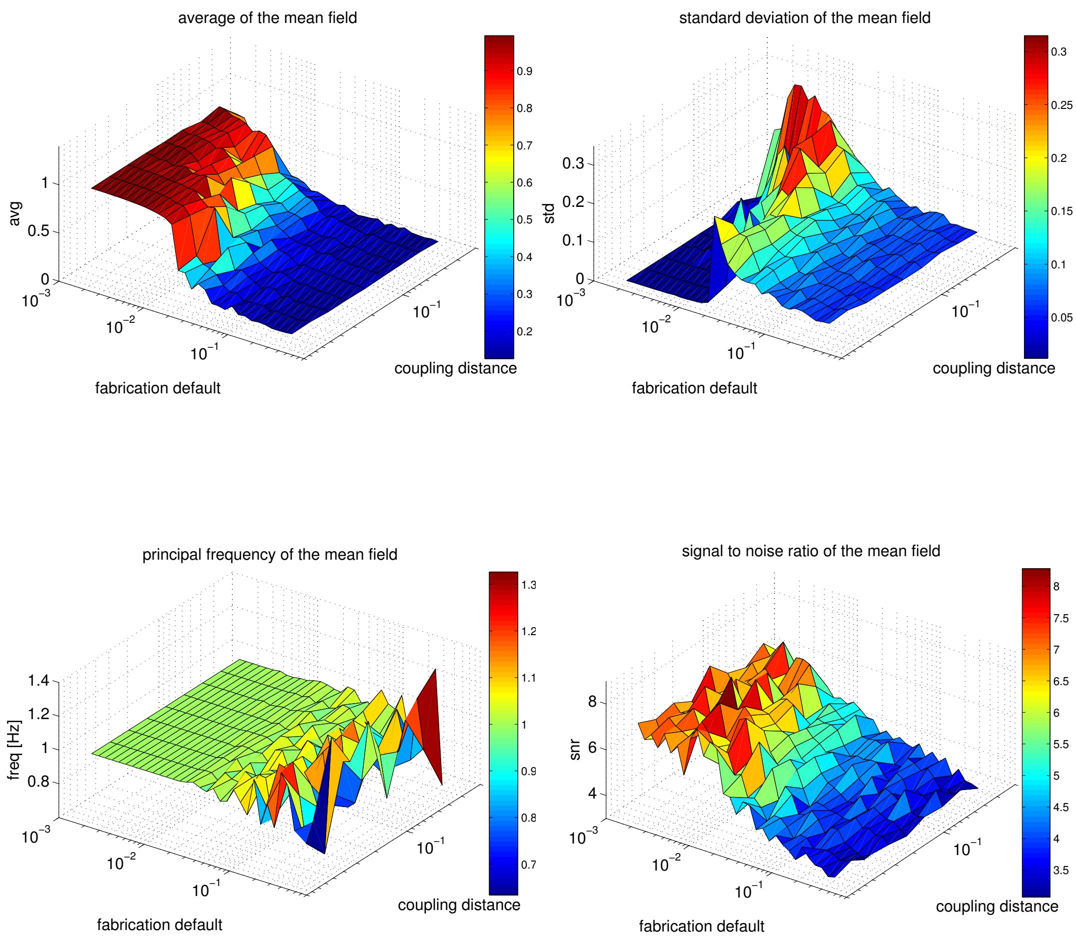


Figure 3.11: Impact of the coupling distance $d_{coupl} \in [3 * 10^{-2}; 8 * 10^{-1}]$ in presence of fabrication's default $P_{fd} \in [1 * 10^{-3}; 8 * 10^{-1}]$. The level of fabrication's default leading to the phase transition between the synchronized state and the non-synchronized state, is dependent on the distance of coupling. A next neighbourg coupled ($d_{coupl} = 3 * 10^{-2} <=> 1 oscillator$) population is lesser robust than a globally coupled one.

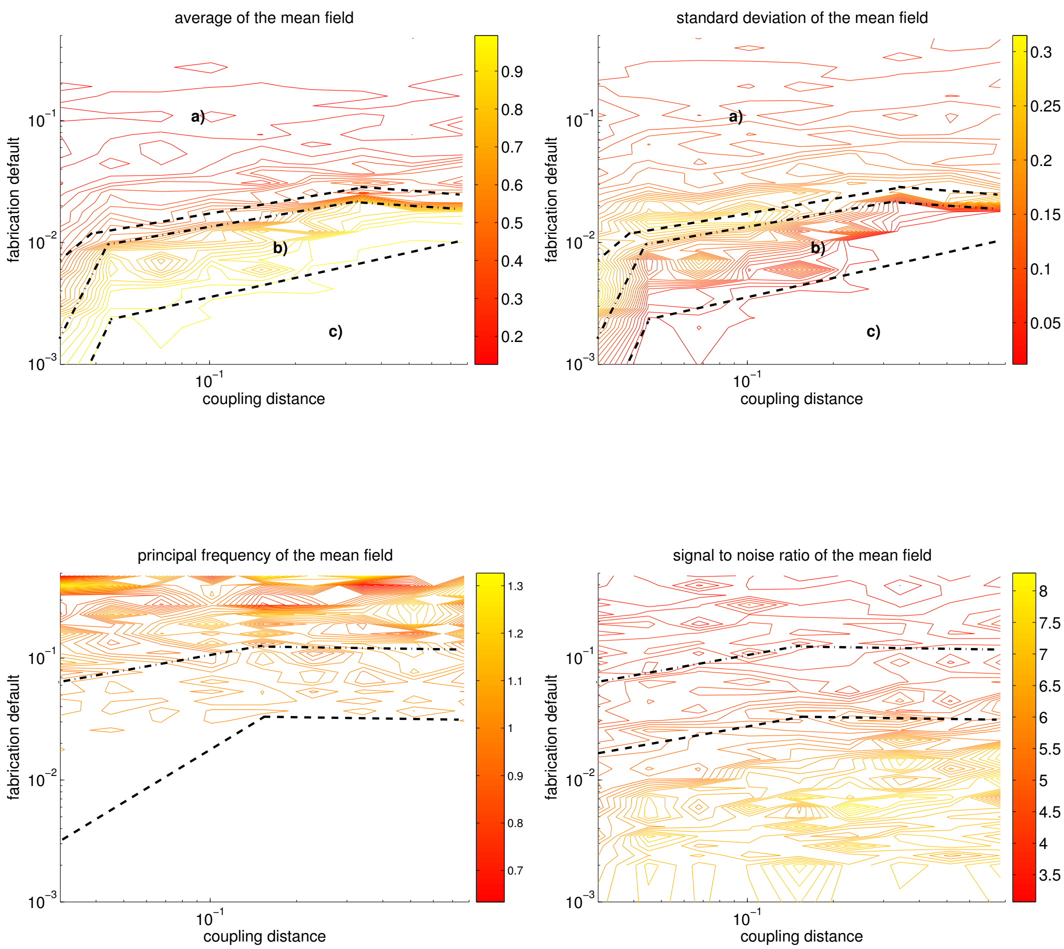


Figure 3.12: Impact of the coupling distance $d_{coupl} \in [3 * 10^{-2}; 8 * 10^{-1}]$ in presence of fabrication's default $P_{fd} \in [1 * 10^{-3}; 8 * 10^{-1}]$. On the two upper plots, c) is the region of synchronization, b) the transition region and a) the region of non-synchronizion. The dotted-dashed line corresponds to 50% of synchronization. The transition region (b) corresponds to the region of maximal standard deviation. The maximal system's marge to fabrication's default is dependent on the distance of coupling. Indeed, for next neighbor coupling ($d_{coupl} = 3 * 10^{-2} <=> 1 oscillator$), the phase transition occurs for $P_{fd}^{max} < 1 * 10^{-3}$ and for a distance of coupling equal to $d_{coupl} = 0.5$, the phase transition occurs when $P_{fd}^{max} > 8 * 10^{-3}$. The relation between fabrication's default and the distance of coupling seems to be that until the distance of coupling is bigger than $d_{coupl} \approx 1.5 * 10^{-1}$, there is no significant increase of robustness (i.e $P_{fd}^{max} < 3 * 10^{-2}$, but for smaller distance of coupling the marge decreases, reaching $P_{fd}^{max} < 3 * 10^{-3}$ for $d_{coupl} = 3 * 10^{-2}$ (next neighbour coupling) (cf. plot of the principal frequency).

Impact of component aging

Fig. 3.13 and fig. 3.14 describes the system's behaviour with the aging of the components, for different values of coupling strength. The system's ability to be robust to the effects of the aging of the components increases, when the coupling gets stronger, the maximal value for component's aging parameter, to keep the system synchronized, increases too. This maximal value going from $P_{cpa}^{max} < 5 * 10^{-6}$ for $\lambda = 1 * 10^{-2}$, to $P_{cpa}^{max} < 1 * 10^{-4}$ for $\lambda \approx 1$ (cf. fig. 3.14). This gain in robustness, does not increase the frequency stability but improves the signal to noise ratio (cf. fig. 3.14). To obtain a gain in frequency stability, the change of the geometrical disposition (annulus to torus) seems to be the only way (cf. fig. 3.15). Indeed, the study of the effects of the population's size (cf. fig. 3.16) does not show a significant change of the robustness of the system, and the distance of couplinf influences only the signal to noise ratio (cf. fig. 3.17). To summarize, to increase robustness against the aging of the components it is recommended to use a population in two dimensions with the strongest coupling, before oscillation deapth ($\lambda < 1$) and the maximum distance of coupling. For a value of component aging$P_{cpa} > 1 * 10^{-4}$, the processing time of the simulation explodes, so unfortunately the simulation for these parameter's values has not been made. The system's marge to component agingis hard to determine, with the result I obtained. Indeed, the marge is always different for the four simulations I do on this parameter.

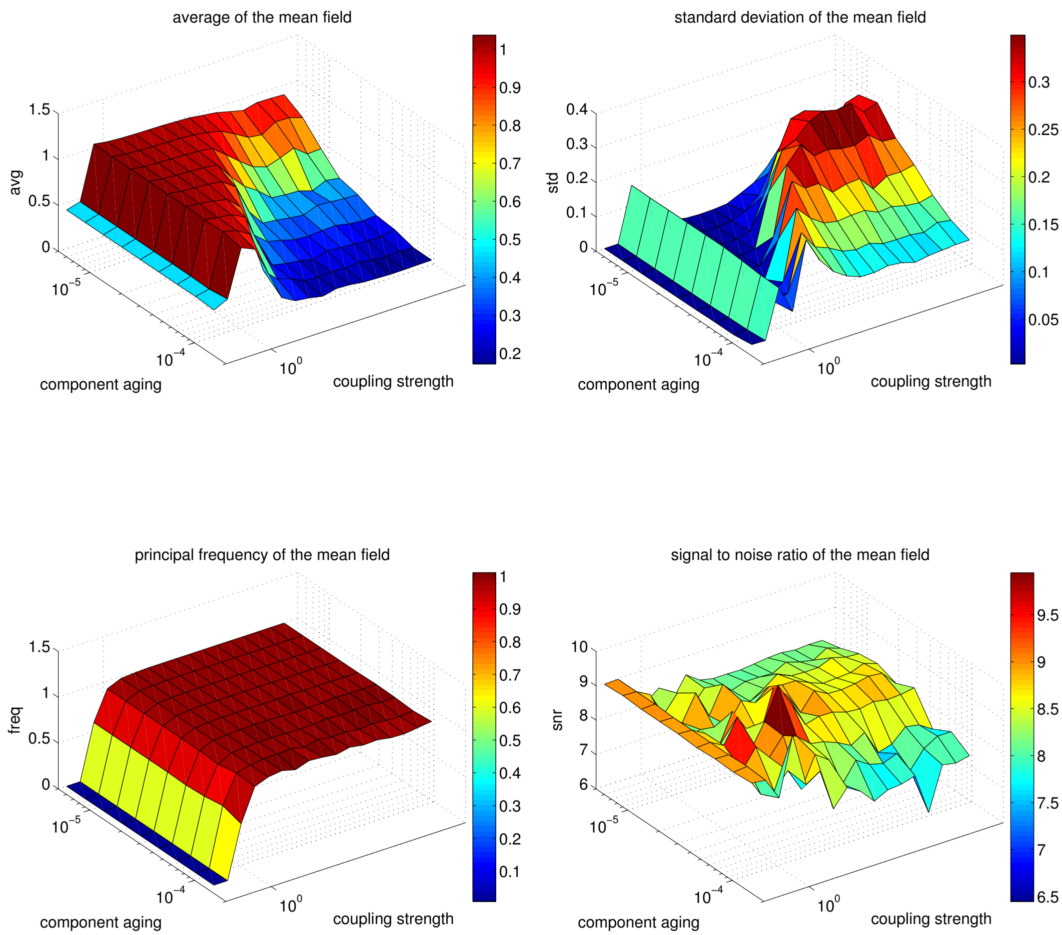


Figure 3.13: Impact of the coupling strength $\lambda \in [1 * 10^{-4}, 10]$ in presence of component aging $P_{cpa} \in [5 * 10^{-6}, 1 * 10^{-4}]$. These four plots, shows the phase transition between synchronized state (avg=1) and non-synchronized state (avg≈0) resulting from the increase of the level of component's aging. The system's marge to component agingvaries in function of the coupling strength. Indeed, one can see that for a weak coupling $\lambda < 1*10^{-2}$, a level of component agingequal to $P_{cpa} = 1 * 10^{-5}$ is sufficient to prevent the system from synchronization. For a stronger coupling, $\lambda > 1 * 10^{-1}$ this level of component aging$P_{cpa} = 1 * 10^{-5}$ is no longer perturbative.

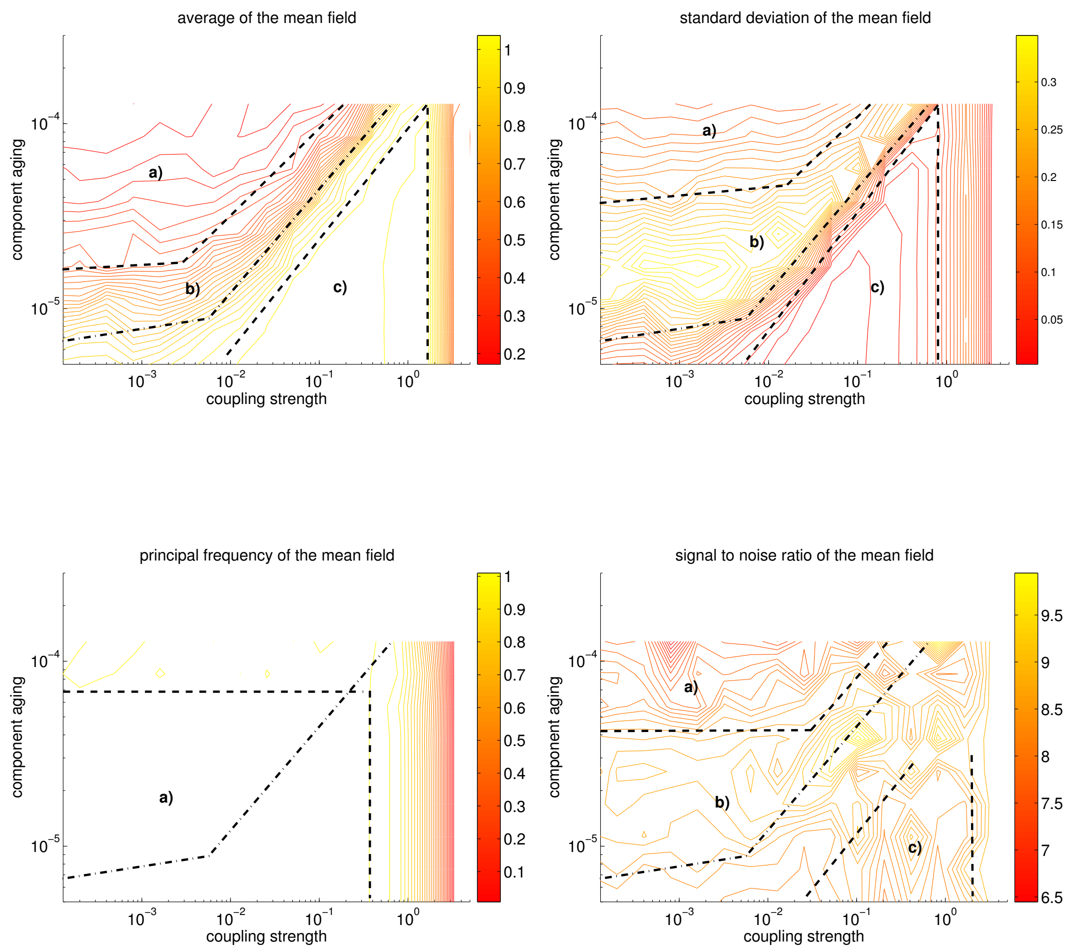


Figure 3.14: Impact of the coupling strength in presence of component aging $P_{cpa} \in [5 * 10^{-6}, 1 * 10^{-4}]$ $\lambda \in [1 * 10^{-4}, 5]$ The axes are volontary too big, to help the comparison with the case in two dimension (cf. fig.3.15). On the four plots, the dotted-dashed line corresponds to 50% of synchronization. System's marge to component agingincreases when the coupling strength gets stronger, this do not imply changes on frequency stability but only increases the signal to noise ratio. For a weak coupling, $\lambda < 1 * 10^{-2}$, the synchronization is already no more complete (avg¡1), even for a little component agingvalues ($P_{cpa} < 5 * 10^{-6}$). Then for coupling strength value between $\lambda \in [1 * 10^{-2}, 1.5]$, the system's marge to component agingincreases from $P_{cpa} \approx 5 * 10^{-6}$ for $\lambda < 1 * 10^{-2}$, to $P_{cpa} \approx 1 * 10^{-4}$ for $\lambda \approx 1$.

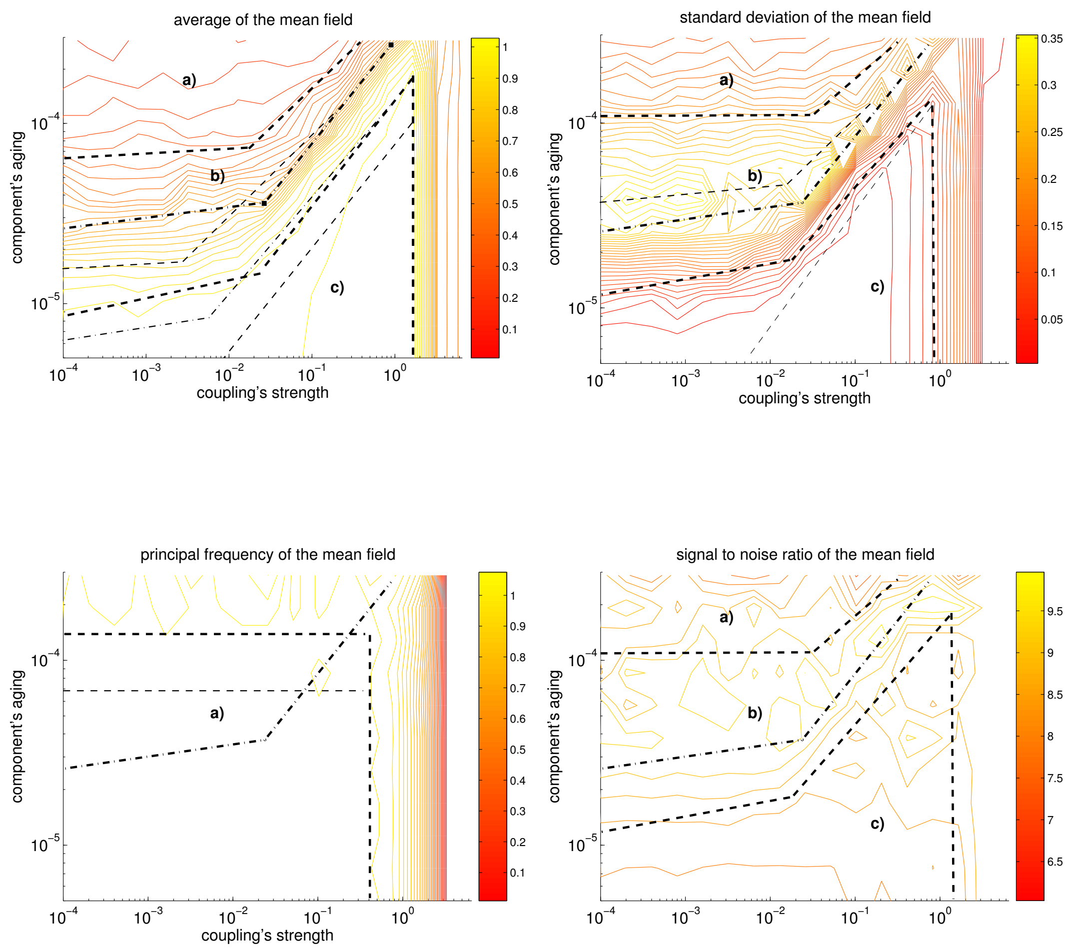


Figure 3.15: Impact of the coupling strength in presence of component aging $P_{cpa} \in [5 * 10^{-6}, 3 * 10^{-4}]$ $\lambda \in [1 * 10^{-4}, 5]$ These four plots shows the increase, of the system's robustness to effects of the aging of the components, due to the change from a population in one dimension (annulus) to a population in two dimension (torus). Indeed, going from one to two dimension, increases the synchronization domain (c in avg plot), the frequency stability and enhances the signal to noise ratio.

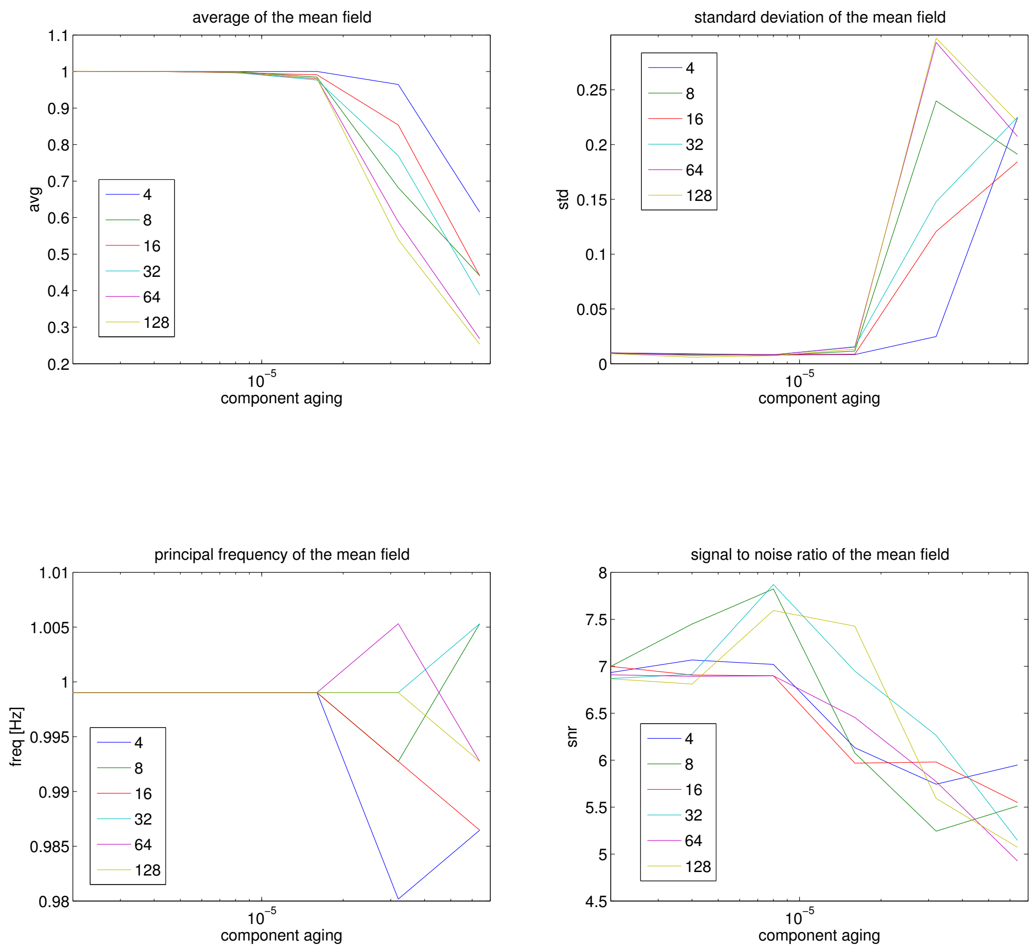


Figure 3.16: Impact of the number of oscillators $n < in[4; 128]$ in presence of component aging $P_{cpa} \in [2 * 10^{-6}; 7 * 10^{-5}]$. Going from 4 to 128 oscillators corresponds not to a real enhancement of the quality of the signal, unless maybe the frequency degradation gets slower (2% to 1%) for $P_{cpa} = [4*10^{-5}$. But more than 32 could be advised.

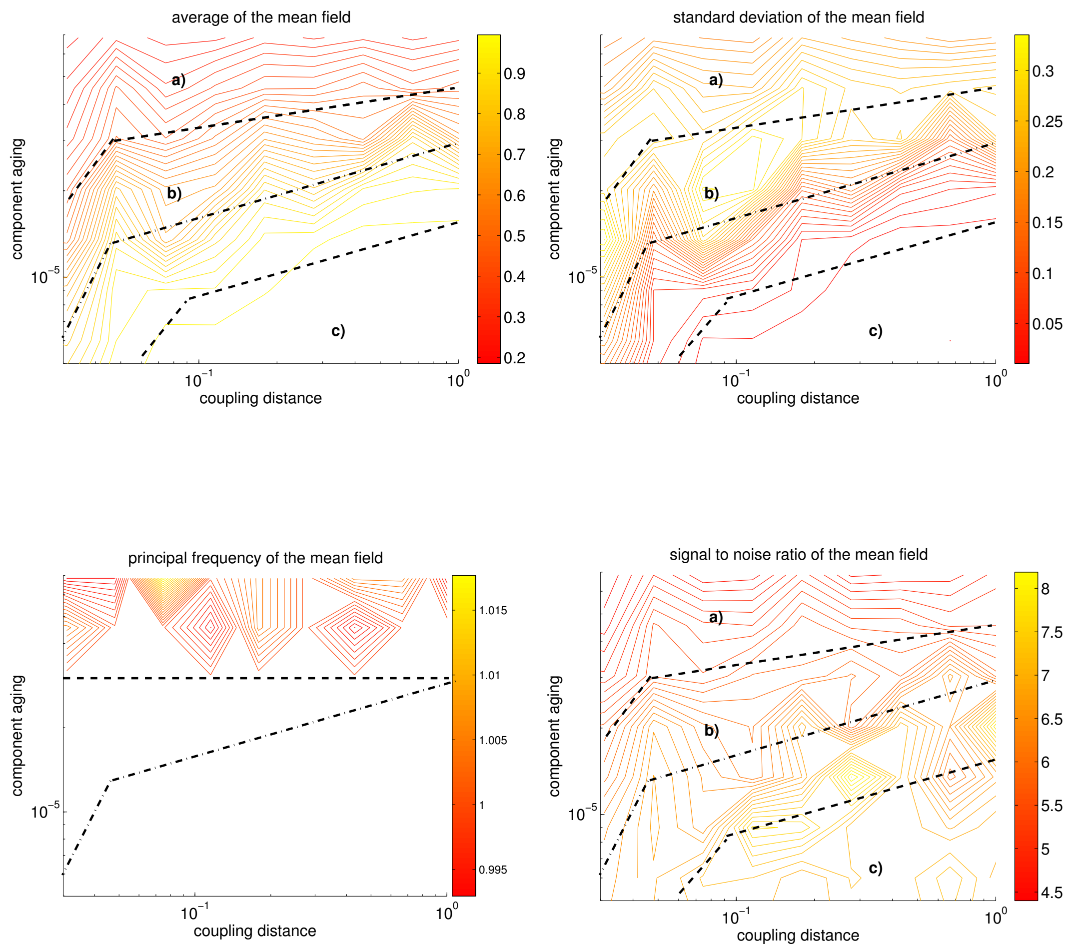


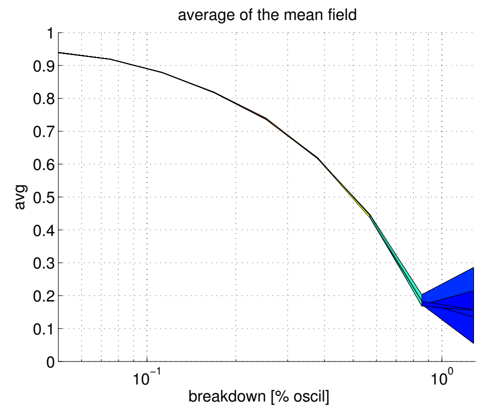
Figure 3.17: Impact of the coupling distance $d_{coupl} \in [3 * 10^{-2}; 1]$ in presence of component aging $P_{cpa} \in [4 * 10^{-6}; 7 * 10^{-5}]$. On the plots, c) is the region of synchronization, b) the transition region and a) the region of non-synchronizion. The dotted-dashed line corresponds to 50% of synchronization. The transition region (b) corresponds to the region of maximal standard deviation. The increase of the distance of coupling leads to an enhancement of the robustness of the system in regard to synchronization and signal to noise ratio, but do not increases the frequency stability. The frequency remains stable for a value of component aging $P_{cpa} = 3 * 10^{-5}$, for any distance of coupling. But the more the distance of coupling increases, the more the signal to noise ratio (snr) remains high: on dashed-dotted line on snr plot snr=7 all along. So for $d_{coupl} = 1 * 10^{-1}$ and $P_{cpa} = 2 * 10^{-5}$: snr$\approx$ 6, and for $d_{coupl} = 1$ and $P_{cpa} = 2 * 10^{-5}$: snr$\approx$ 7.

Impact of breakdowns

As described in section 2.4.5, three kinds of breakdowns have been simulated. The first one is : when an oscillator breaks he keeps its values constant, it stops. When this kind of breakdown occurs the system starts loosing synchronization proportionnaly to the percentage of broken oscillators and its signal to noise ratio gets down too (c.f fig. 3.18). The frequency is always equal to one until there is more broken oscillators than not broken ones and then the frequency is equal to zero (a constant signal (broken oscillator) has a frequency equal to zero). The distance of coupling seems not to influence the system's behavior, nevertheless next neigbourg coupling ($d_{coupl} = 0.02$) is worst in regard to signal to noise ratio (c.f fig. 3.19).

The second kind of breakdown is : when an oscillators breaks he becomes a generator of noise. For this kind of breakdown, the system's reaction is a progressive degradaton of the synchronization, exactly similar when the oscillators keeps a constant value (first kind of breakdown) but a slower degradation of the signal to noise ratio (c.f fig. 3.20). The frequency remains stable until there is just one or two not-broken oscillators left. Fig. **??** shows there is no relation between robustness to "noise generator" breakdown and the distance of coupling.

The last kind of breakdown is breakdown of coupling, to test system's tolerance to breakdowns of coupling it is necessary to add a little source of perturbation, if not, there is no degradation of the system to observe. Indeed the breakdown of coupling is not enougth perturbative, that there is no degradation to observe for a non-perturbed system. Breakdowns on coupling seems, instead of all other breakdowns, to be dependent on distance of coupling. The principal frequency of the mean field is stable for a distance of coupling $d_{coupl} > 0.1$ until the percentage of breakdowns exceeds thirty percent. In fact, the distance of coupling make increase the general robustness of the system : synchronization is kept more longer, for a level of fabrication's default of 0.003, synchronization is of 30% for globally coupled $d_coupl = 1$ and of 10% for $d_coupl = 0.3$ (i.e 30%) (cf. fig.3.22). Fig.3.23 shows, the effects of coupling strength according to the percentage of breakdowns of coupling. This figure tends to proof that since the strength of coupling $\lambda = 5 * 10^{-3}$, the system becomes more and more robust. For simulation of fig.3.23, the little source of pertubation added is a default of fabrication $P_{fd} = 0.003$, the perturbation is so little that the system's frequency do not change even if all coupling are broken.



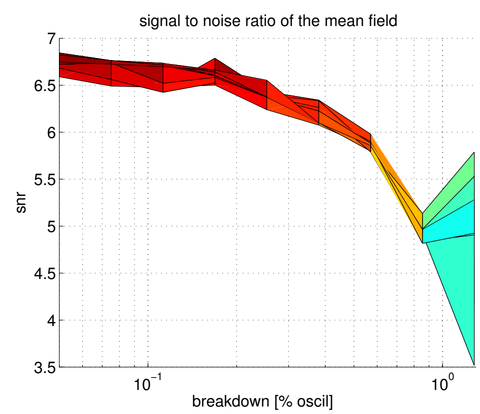


Figure 3.18: Degradation of the synchronization in presence of a the percentage of breakdown NOISE CST with $R_{bkd} \in [5*10^{-2}; 1.8]$ and f_{bkd}=0.05. These plots shows the linear law between the percentage of oscillators broken (ie.R_{bkd}) and the average of the mean field (ie. percentage of synchronization). One can see that at a fixed R_{bkd} the average equal 1-R_{bkd}. The signal to noise ratio decreases to form ≈ 6.7 to ≈ 4.7 (for $R_{bkd} = 0.9$).

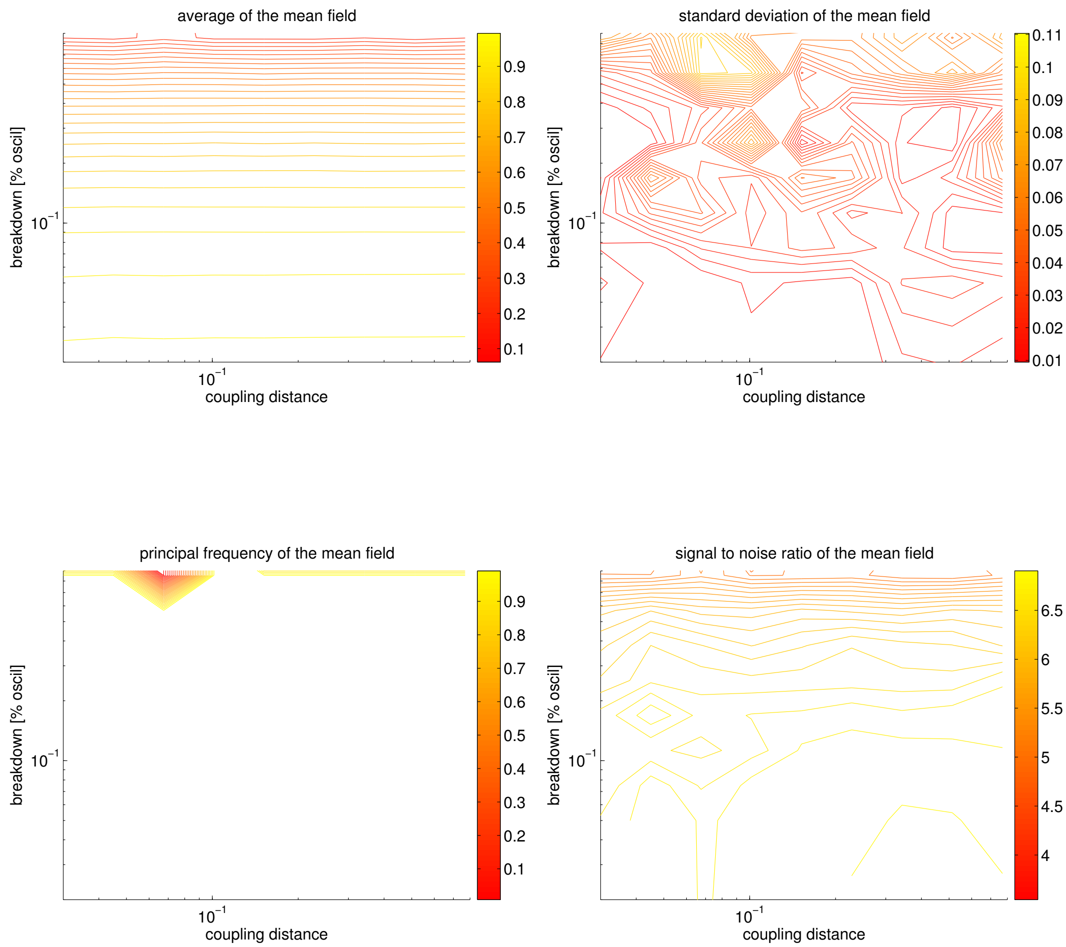
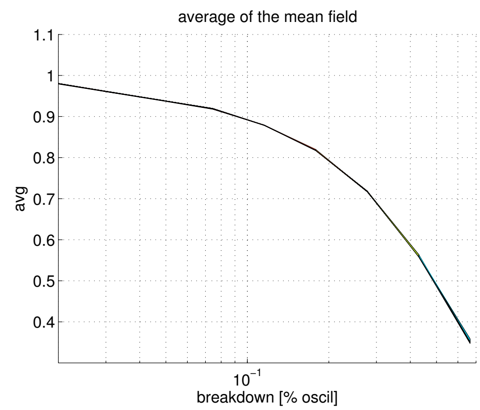


Figure 3.19: Impact of the coupling distance $d_coupl \in [3 * 10^{-2}; 0.8]$ in presence of a the percentage of breakdown CST $R_{bkd} \in [5 * 10^{-2}; 1.8]$ and f_{bkd}=0.05. The plot of the average confirms the linear degradation in fonction of the percentage of breakdown. With a weak coupling $d_coupl < 3*10^{-1}$, the standard deviation is more weak, for small percentages of breakdowns, than for a stronger coupling.



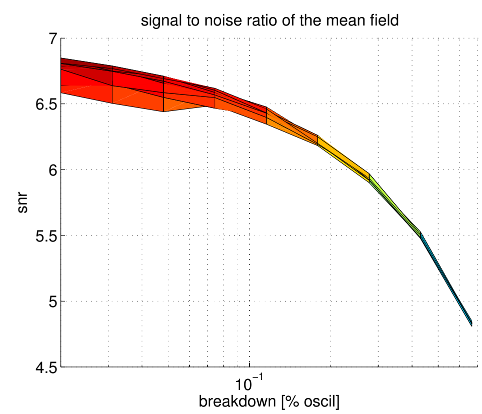


Figure 3.20: Degradation of synchronization in presence of a the percentage of breakdown NOISE GENERATOR $R_{bkd} \in [2 * 10^{-2}; 0.7]$ and f_{bkd}=0.05. The degradation is linear, actually, a percentage of broken oscillators R_{bkd} means an average of the mean field equal to 1-R_{bkd}. The signal to noise is degradate too from ≈ 6.7 to ≈ 4.7

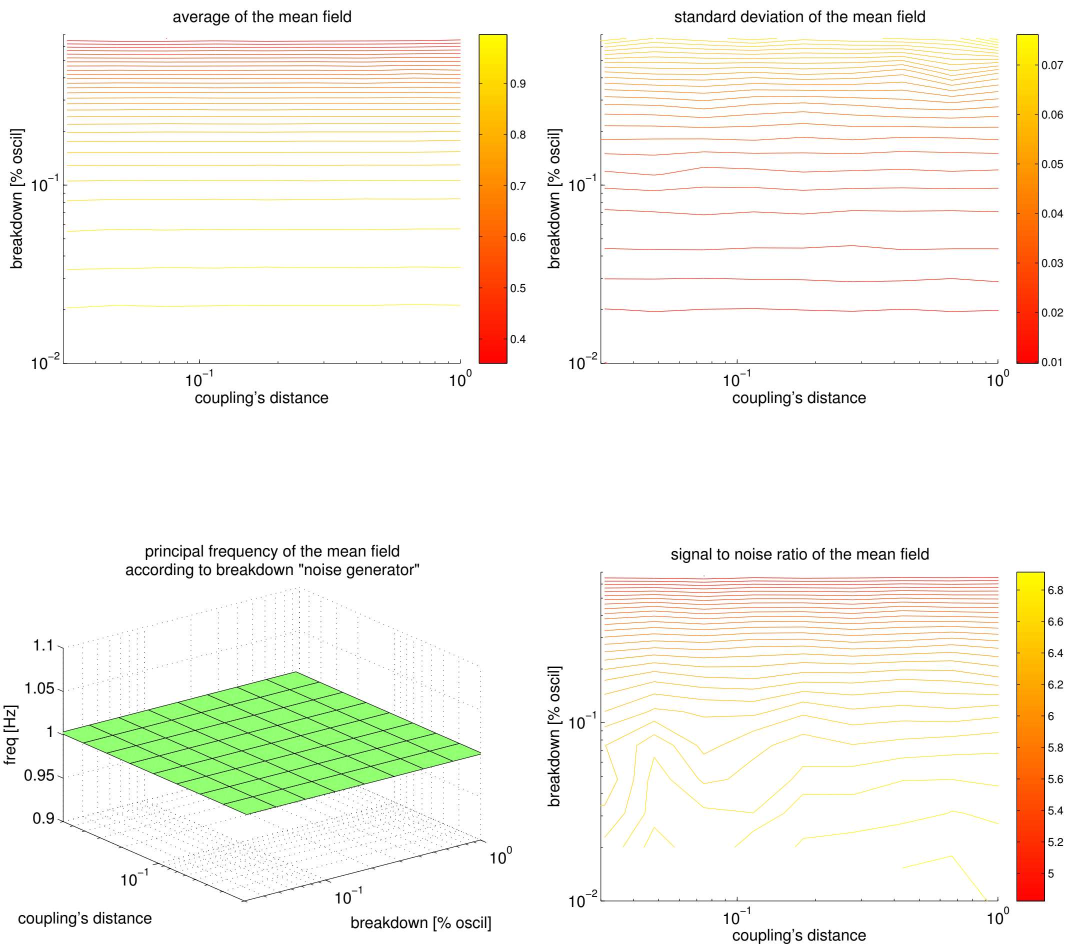


Figure 3.21: Impact of the coupling distance $d_c oupl \in [3*10^{-2}; 1]$ in presence of a percentage of breakdown NOISE GENERATOR $R_{bkd} \in [2 * 10^{-2}; 0.7]$ and f_{bkd}=0.05. The frequency remains stable, even the average of the mean field decreases, only signal to noise ratio tends to prooves that a coupling stronger than $d_c oupl = 2 * 10^{-1}$ is better for the robustness of the system.

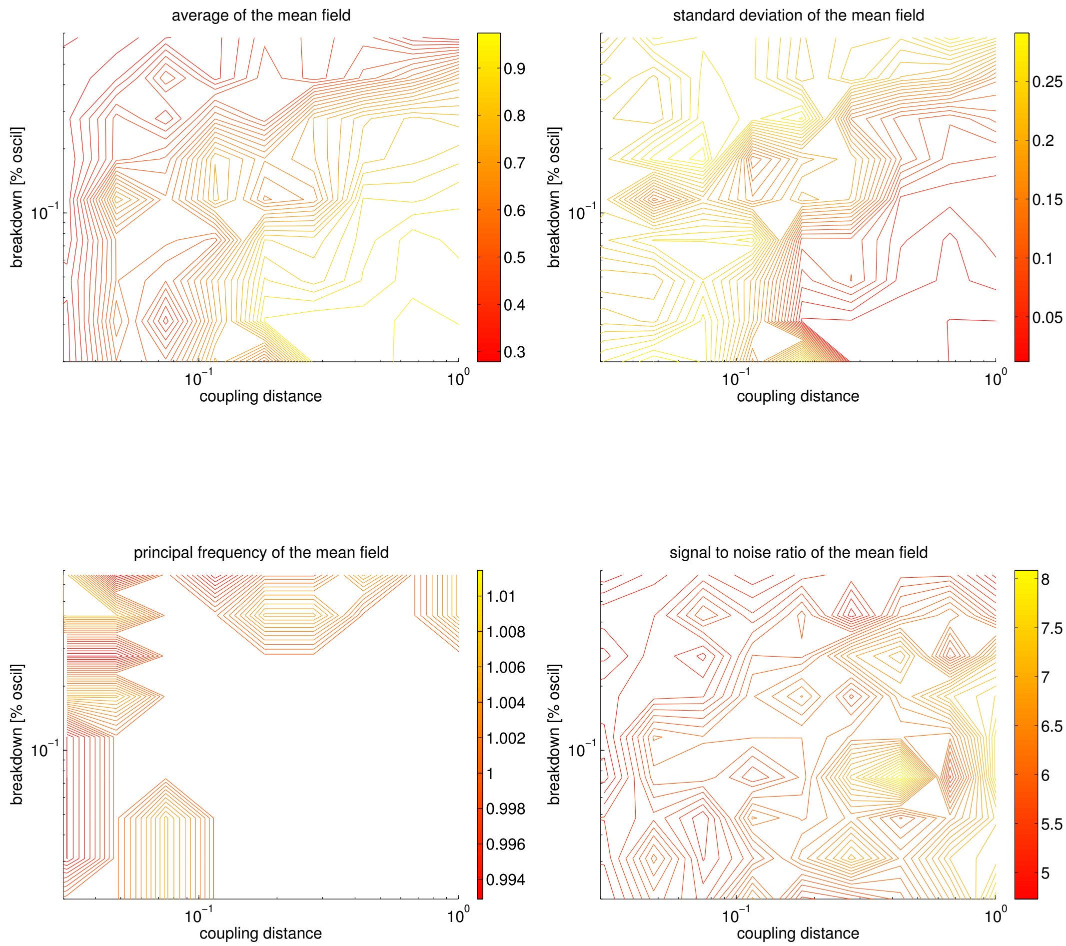


Figure 3.22: Impact of the coupling distance $d_coupl \in [3*10^{-2}; 1]$ in presence of a the percentage of breakdown COUPLING $R_{bkd} \in [2*10^{-2}; 0.7]$ and f_{bkd}=0.05 and fabrication default = 0.01 (fabrication default is included to see a degradation of the frequency). Breakdowns on coupling seems, instead of all other breakdowns, to be dependent on distance of coupling. The principal frequency of the mean field is stable for a distance of coupling $d_{coupl} > 0.1$ until the percentage of breakdowns exceeds thirty percent. In fact, the distance of coupling make increase the general robustness of the system : synchronization is kept more longer, for a level of fabrication's default of 0.003, synchronization is of 30% for globally coupled $d_coupl = 1$ and of 10% for $d_coupl = 0.3$ (i.e 30%).

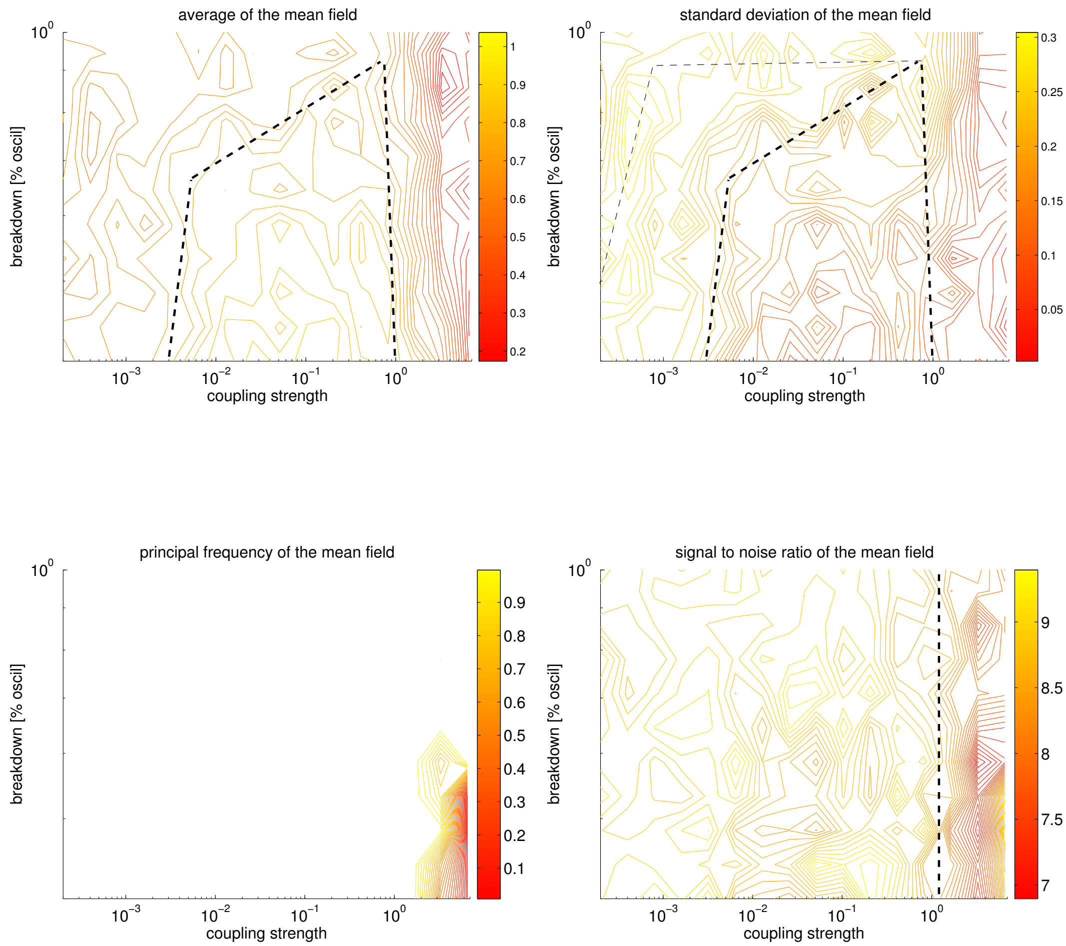


Figure 3.23: Impact of the coupling force in presence of a the percentage of breakdown COUPLING $R_{bkd} \in [2 * 10^{-2}; 0.7]$ and f_{bkd}=0.05 and $P_{fd} = 0.003$. These plots tends to proof that since the strength of coupling $\lambda = 5 * 10^{-3}$, the system becomes more and more robust. The little added source of pertubation, is a default of fabrication $P_{fd} = 0.003$. The perturbation is so little that the system's frequency do not change even if all coupling are broken.

3.3 Second experiment : A robust square wave generator

The idea is to obtain, achieving high-order in-phase synchronization between two (or more) clusters of oscillators, a square wave generator (cf. sec. 1.3.1). The Fourier serie of a square wave being :

$$\frac{4}{\pi} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n} sin(\frac{n\pi x}{L}) \tag{3.2}$$

with L the half-period of the square wave. To approximate the square wave, a population of oscillators separate in two clusters is used. This bi-clustered population of oscillators with, in the first cluster an initial frequency $\omega_{01} = 1$ and, in the second, an initial frequency $\omega_{02} = 3$, has a mean field equal to a square wave (when the system is synchronized). In order to include the term of normalization of the fourier serie ($\frac{1}{n}$), the first cluster of oscillators contains three times more oscillators than the second cluster.

3.3.1 Performance measurement

To mesure the performance of the system, the mean field is compared to a perfect square wave of amplitude =0.6 (this is the amplitude obtained with the unperturbed system). The error in terms of least square is calculated and its average and standard deviation along time are calculated too. Contrary to the sinus wave generator, the population is synchronized when the average (of error) is minimal. When five cluster were used, tests made during this work demonstrate that a not perfect's keep of frequency (for the cluster) is not the principal source of error. Actually the most important constraint to have a square wave is to have a sum of perfectly in-phase sinus. This is why the stability of the two frequencies is not tested.

3.3.2 Results

Many results have been generated to understand the behaviour of a clustered population of oscillators. At the beginning a population with five clusters was used, giving a more precise square wave. But it was very hard to achieve and maintained high-order synchronization between the high frequency cluster (5,7,9), so it has been decided to use only two clusters to generate the square wave. Fig. 3.24 shows different square wave obtained according to the level of noise on initial condition. This is a simple manner to test the property of synchronization of a system, the time of synchronization. Actually if the initial condition are to much different (cf. fig. 3.24 case b), the mean field, after 400[sec] is still not a square wave. The robustness properties can increase if the coupling gets stronger. For the square wave only robustness to thermal noise (cf. fig.3.26) and robustness to initial condition (cf. fig. 3.25), are presented. These two figures shows that, the properties of robustness demonstrates for the sinus wave generator are not implicitely extended to the square wave generator. Actually high-order sychronization is harder to maintain.

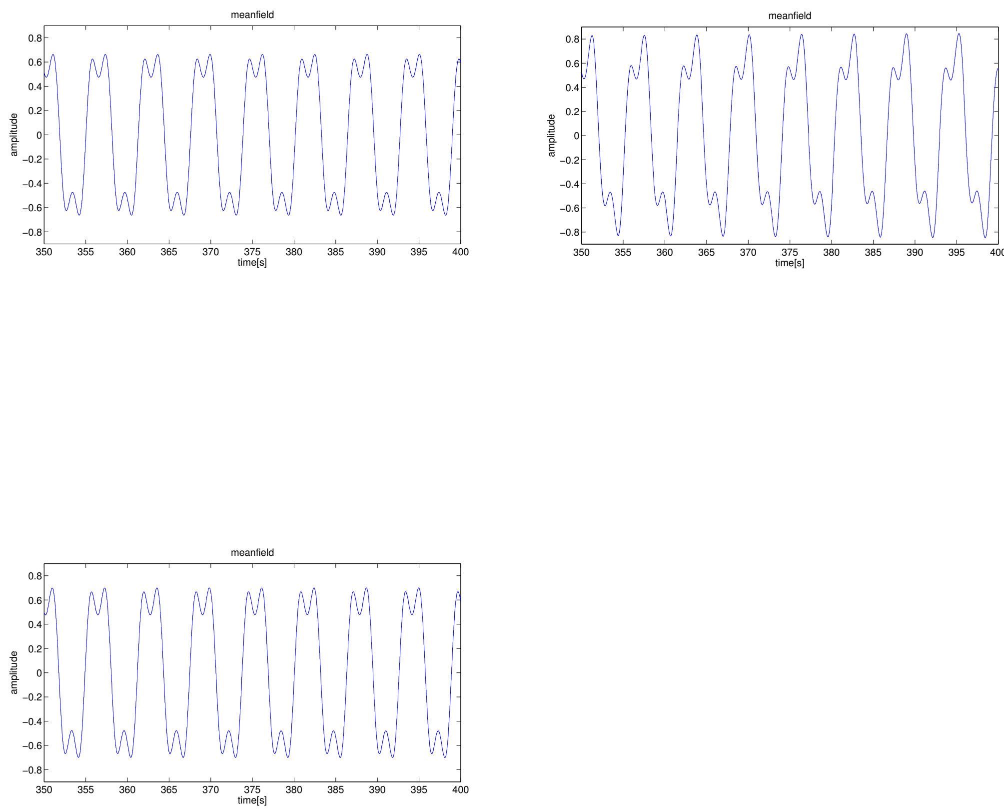


Figure 3.24: a) top-left plot : noise on initial condition variation = 0.8 strength of coupling = $1 * 10^{-3}$ b) top-right plot : noise on initial condition variation = 1.3 strength of coupling = $1 * 10^{-3}$ c) bottom-left plot : noise on initial condition = 1.3 strength of coupling = $1 * 10^{-1}$ Increasing the strength of coupling let the system to synchronize with a more random initial conditions.

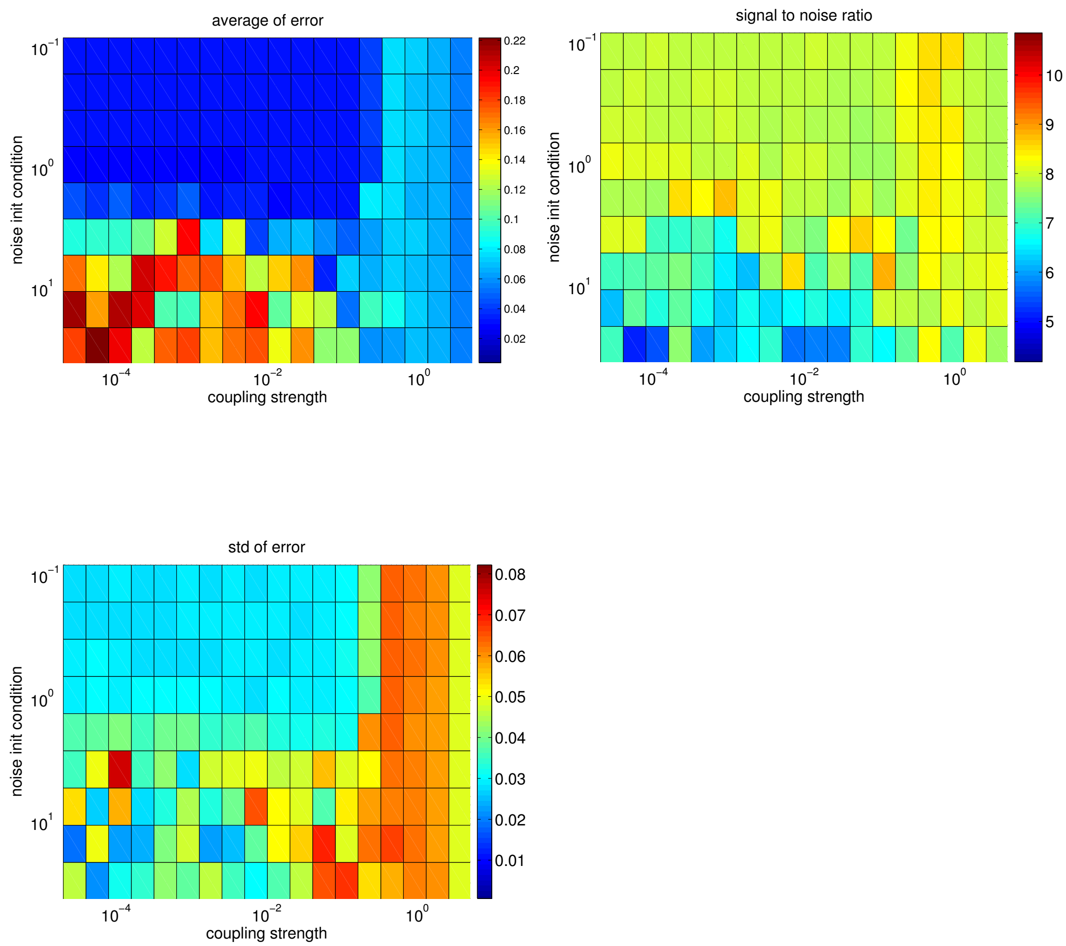


Figure 3.25: Impact of coupling strength $\lambda \in [1*10^{-5}; 6]$ in presence of noise on initial conditions$\in [1*10^{-5}; 6]$. One can see that for too strong coupling (i.e $\lambda > 1*10^{-1}$), the system is no more able to keep a perfect square anf start loosing precision. Indeed, signal to noise ratio raises up, standard deviation and average of error too. But just before, $\lambda = 1*10^{-1}$, there is a little enhancement of the robustness, and a noise on initial condition equal to 1.3 is no more perturbative.

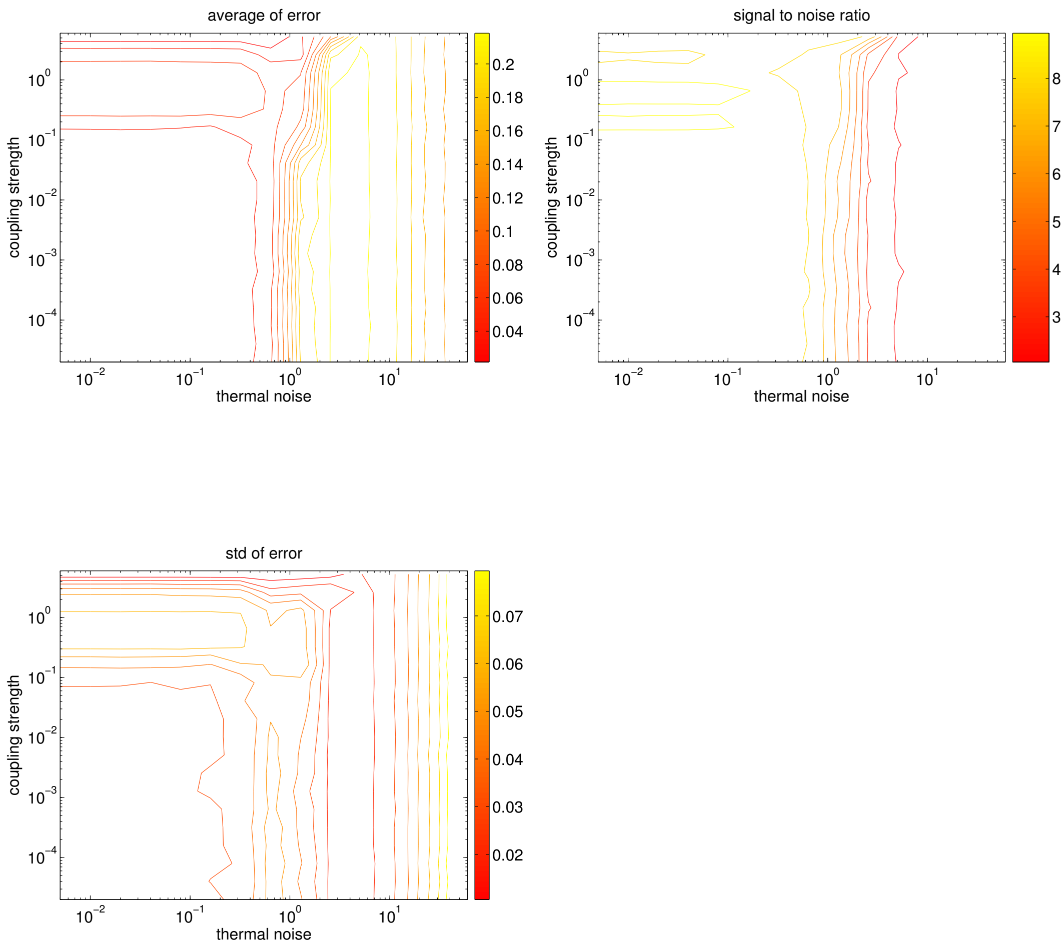


Figure 3.26: Impact of coupling strength $\lambda \in [1 * 10^{-5}; 5]$ in presence of thermal noise $P_{thn} \in [1 * 10^{-5}; 6]$. Increasing the strength of coupling leads to an enhancement of the robustness of the system. As one can see, for a strength of coupling $\lambda < 5 * 10^{-2}$ the maximum of thermal noise before the beginning of system's degradation takes place when $P_{thn} > 6 * 10^{-1}$. And for a strength of coupling $\lambda = 5 * 10^{-1}$ (strong coupling) the maximum is now $P_{thn} \approx 1$.

3.4 Concluding words

3.4.1 About the project

All the properties observed, in the case of the sinus wave generator, about the robustness are summarized in table 3.2. These properties show clearly that, in all cases, a stronger coupling increases the robustness of the system. It could be clear that increasing the strength of interactions between the oscillators, leads to less freedom for this oscillators and so if an oscillator is perturbed, it will return on the right way faster. But in the other way, increasing the strength of coupling, come down to saying that a perturbation is more diffused in all population and so has more general impact. This is why this property was not obligatorily expectable. The distance of coupling plays also an important role in the system. As observed in experiments, in the case of fault of fabrication, component aging and breakdowns on coupling, it is advised to use a population of coupled oscillators, coupled more than just next neighbor. As hoped, a population do not need to have a huge number of oscillators to be robust. In most case (fault of fabrication, thermal noise, component aging), a large population (about 100 oscillators) seems not to be more robust. In fact, the dimension of the population has clearly a bigger effect on robustness. Actually, without changing the size of the population, just passing to a two dimensional population, enhances the robustness.

Globally, the results of the two experiments, sinus and square wave generator, tend to confirm the hypothesis : Synchronization has really intrinsic properties of robustness. Indeed a population of oscillators proves to be robust to thermal noise, fabrication defaults and component aging, moreover it presents an real potential against breakdowns. Actually, the degradation is linear with the percentage of broken oscillators, which is a different from a total degradation for one breakdown, like in usual systems (computer, electronically circuits,...).

3.4.2 Future investigations

I have presented no results with the Van der Pol oscillator, but during the work I have, despite everything, made several simulation with Van der Pol oscillators. All these simulations tended to prove that the same general behavior is expectable with this other kind of oscillators. So in a future work it will be interesting to extend the results to population of Van der Pol oscillators. An other subject to be interesting is the traveling waves. The program implemented for this work allows to configure a population able to create traveling waves. This work don't speak about traveling waves at all, but it was also an idea of application, like the sinus wave generator. Traveling wave are well study in literature about complex systems, so in a

to enhance the robustness to	best configuration
thermal noise	strong coupling and about 40 oscillators
fault of fabrication	strong coupling, more than 4 oscillators, a two dimensional population and a big coupling distance
component aging	strong coupling a two dimensional population and a big coupling distance
variation of initial conditions	strong coupling
breakdowns on coupling	strong coupling and a big coupling distance

Table 3.2: Table to summarize the observations made during this work about the robustness of a coupled population of oscillators (this table do not take into account the square wave generator). For each perturbations, an increase of the system parameter implies either

future it will be interesting to extend this work to the study of traveling waves.

3.4.3 Acknowledgments

I would like to thank Jonas Buchli, my supervisor, for all the councils he gave me, about how a scientific work must be done. It has been also a great technical and theoretical support.

I would like also to thank the BIRG, and more precisely Auke Jan Ijspeert, for letting students doing such work in the so interesting world of complex systems.

And finally, I thank Marc-Antoine Nuessli and Thibault Gennessay for interesting discussions we have had during this project.

Bibliography

[1] A. Pikovsky, M. Rosenblum and J. Kurths *Synchronization, A universal concept in non-linear sciences,* (Cambridge University Press, Cambridge, UK, 2001).

[2] A. Pikovsky, M. Rosenblum *Controlling Synchronization in an Ensemble of Globally Coupled Oscillators,* (Department of Physics, University of Postdam, Postdam, 2004).

[3] J. Buchli, J. Ijspeert *Distributed central pattern generator model for robotics application based on phase sensitivity analysis,* (LSL, Swiss Federal Institute of Technology, Lausanne, 2003).

[4] Wolfram Research MathWorld *web site on mathematics,* (http://mathworld.wolfram.com/).

[5] Strogatz S. *Non-linear Dynamics and Chaos,* (Addison Wesley Publishing Company).

[6] Strogatz S., Stewart I. *Coupled Oscillators and Biological Synchronization,* (Scientific American , december 1993).

[7] J. Ijspeert, J. Nakanishi, Schaal S. *Movement Imitation with Nonlinear Dynamical Systems in Humanoid Robots,* (Computational Learning and Motor Control Laboratory, University of Southern California, Los Angeles).

[8] J. Ijspeert, J. Nakanishi, Schaal S. *Learning Rhytmic Movements by Demonstration using Nonlinear Oscillators,* (Dept. of Computer Science, University of Southern California, Los Angeles, 2002).

[9] Rubin J., Terman D. *Analysis of Clustered Firing Patterns in Synaptically Coupled Network of Oscillators,* (Dept. of Mathematics, The Ohio State University, Columbus).

[10] Kunz H., Ackermann P. *Simulation of circadian rhythm generation in the suprachiasmic nucleus with locally coupled self-sustained oscillators,* (Inst. of Pharmacology and Toxicology, University of Zurich, 2003).

[11] Daido H. *Quasientrainment and Slow Relaxation in a Population of Oscillators with Random and Frustrated Interactions,* (Dept. of Physics, Kyushu Institute of Technology, Tobata, 1991).

[12] Distrelec *web site on electonical material,* (http://www.distrelec.ch/).

For any question on this work call the author at `jerome.guerra@epfl.ch`